Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints

Y.-P. Lee

R. Pandharipande

Preliminary Version
31 July 2004
Please send corrections & comments to the authors

Contents

Part 2

Quantization and Virasoro

In Part 2, Givental's formula for the τ-function of a calibrated, semisimple Frobenius manifold is presented and the Virasoro constraints are derived. Givental's formula for descendent potentials is most naturally written in terms of quantized symplectic transformations. The quantization formalism is reviewed in Chapter 8. In Chapter 9, Givental's formula for the τ-function is presented after a discussion of R and J-calibrations. Properties of the associated higher genus potentials are studied in Chapter 10. Finally, the Virasoro constraints are derived in Chapter 11.

CHAPTER 8

Quantization

1. Overview

Givental's formula for the descendent τ-function of a semisimple Frobenius manifold $\mathcal{F}$ is most naturally written in terms of quantized symplectic transformations. The required quantization formalism is reviewed in Section 2.

Givental's formula is discussed in Chapter 9. The descendent τ-function of $\mathcal{F}$ is proven to satisfy universal Virasoro constraints in Chapter 11. The calculus of quantized operators plays an essential role in the derivation of the Virasoro constraints.

The τ-function of the moduli space of curves was introduced in Chapter 6. Virasoro operators annihilating the τ-function of the moduli space are constructed using the quantization formalism in Section 3.

2. Quantization of infinitesimal symplectic transformations

2.1. The symplectic space $(\mathbb{H}, \Omega)$**.** Let H be a $\mathbb{C}$-vector space of dimension h equipped with a nondegenerate, symmetric, bilinear form $\langle \cdot, \cdot \rangle$. Let $\mathbb{H}$ be the vector space of Laurent polynomials in z with values in H. A canonical nondegenerate symplectic form Ω is defined on $\mathbb{H}$ by

$$\Omega(f(z), g(z)) = \frac{1}{2\pi i} \oint \langle f(-z), g(z) \rangle \, dz, \tag{1}$$

where $f(z), g(z) \in \mathbb{H}$. The pair $(\mathbb{H}, \Omega)$ defines a symplectic vector space.

The symplectic space $(\mathbb{H}, \Omega)$ has a canonical polarization

$$\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_- = T^*\mathbb{H}_+,$$

corresponding to the decomposition of Laurent polynomials into polynomial and polar parts. Both subspaces $\mathbb{H}_\pm$ are Lagrangian by the definition of Ω.

Let $v_1, \ldots, v_h$ be basis of H. We will use the coordinate system,

$$(\gamma^1, \ldots, \gamma^h) = \sum_{i=1}^{h} \gamma^i v_i,$$

defined by the basis. The metric is determined by the matrix,

$$g_{ij} = \langle v_i, v_j \rangle.$$

Let g^{ij} denote the inverse matrix.

Once a basis is chosen for H, Darboux coordinates

$$\{p_k^i, q_k^i\} \tag{2}$$

may be defined for $\mathbb{H}$ with respect to the canonical polarization. The indices of the Darboux coordinates vary in the following sets:

$$k \in \{0, 1, 2, \ldots\}, \quad i \in \{1, \ldots, h\}.$$

Let $\mathbf{p} = \{p_k^i\}$ and $\mathbf{q} = \{q_k^i\}$ denote the coordinate sets.

An H-valued Laurent polynomial $f(z) \in \mathbb{H}$ is written in Darboux coordinates as:

$$f(z) = \sum_{k=0}^{\infty} (p_k^1, \ldots, p_k^h) \frac{1}{(-z)^{k+1}} + \sum_{k=0}^{\infty} (q_k^1, \ldots, q_k^h) z^k.$$

To simplify notation, let p_k and q_k denote the vectors

$$(p_k^1, \ldots, p_k^h), \ (q_k^1, \ldots, q_k^h) \ \in H.$$

For each $f(z) \in \mathbb{H}$, the vectors p_k and q_k vanish for sufficiently large k.

Let $\mathrm{End}(\mathbb{H})$ denote the vector space of linear transformations on $\mathbb{H}$. Linear transformations define vector fields on the symplectic space $(\mathbb{H}, \Omega)$. The minus transformation,

$$f(z) \mapsto f(-z),$$

appears in (1) and will occur often.

Let $A \in \mathrm{End}(\mathbb{H})$. A is an infinitesimal symplectic transformation if

$$\langle (Af)(-z), g(z) \rangle + \langle f(-z), Ag(z) \rangle = 0.$$

Then,

$$\Omega(Af, g) + \Omega(f, Ag) = 0, \tag{3}$$

and A determines a symplectic vector field on $(\mathbb{H}, \Omega)$.

A *quadratic polynomial* P in the Darboux coordinates $\{p_k^i, q_k^i\}$ is a series of homogeneous degree 2 monomials in which each variable occurs only finitely often. There is a bijection between infinitesimal symplectic transformations A and quadratic polynomials P in the Darboux coordinates. Given A, define $P[A]$ by

$$P[A](f) = \frac{1}{2}\Omega(Af, f).$$

The required finiteness for P is an elementary consequence of (3). Viewing A as a vector field on $(\mathbb{H}, \Omega)$, we obtain the relationship

$$dP = i_A\Omega. \tag{4}$$

Conversely, given P, equation (4) defines a unique infinitesimal symplectic transformation $A[P]$.

Geometrically, a symplectic vector field on any symplectic manifold is generated locally by a Hamiltonian function. In our case, A is globally generated by the quadratic Hamiltonian $P[A]$.

A simple example can be found in case H a 1-dimensional vector space. Let A be multiplication by $1/z$. Then, A defines an infinitesimal symplectic transformation, and

$$P[A] = -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1}p_m.$$

Here, coordinates are chosen on H with respect to a basis v_1 satisfying $\langle v_1, v_1\rangle = 1$.

The vector space $\mathbb{A}$ of infinitesimal symplectic transformations is a Lie algebra with bracket defined by the commutator. The vector space $\mathbb{P}$ of quadratic polynomials is a Lie algebra via the Poisson bracket:

$$\{P_1(p,q), P_2(p,q)\} = \sum_{k,i} \frac{\partial P_1}{\partial p_k^i}\frac{\partial P_2}{\partial q_k^i} - \frac{\partial P_2}{\partial p_k^i}\frac{\partial P_1}{\partial q_k^i}.$$

The following Lemma is proven by direct calculation.

LEMMA 1. *A Lie algebra isomorphism,*

$$\mathbb{A} \to \mathbb{P},$$

is defined by the correspondence $A \mapsto P[A]$.

The isomorphism condition is $\{P[A_1], P[A_2]\} = P[[A_1, A_2]]$.

2.2. Quantization. Weyl quantization will be used to to quantize infinitesimal symplectic transformations A via their quadratic Hamiltonians $P[A]$.

The polarization $\mathbb{H} = T^*\mathbb{H}_+$ distinguishes the configuration space $\mathbb{H}_+$ in the phase space $\mathbb{H}$. A quadratic Hamiltonian P determines a classical mechanical system on $\mathbb{H}_+$. Let the Fock space $\mathcal{H}$ consists of functions $\gamma(\lambda, \mathbf{q})$ on $\mathbb{H}_+$ with parameter λ. Quantum mechanical Hamiltonians are quadratic differential operators on $\mathcal{H}$.

Quantization associates quantum mechanical systems to classical mechanical systems. Weyl quantization associates a quadratic differential operator

$$\widehat{P}(q, \frac{\partial}{\partial q})$$

on $\mathcal{H}$, to each quadratic Hamiltonian P.

Consider the following quantization rules in the Darboux coordinates,

$$\widehat{1} = 1, \quad \widehat{p_k^i} = \sum_{i'} \sqrt{\lambda}\, g^{ii'} \frac{\partial}{\partial q_k^{i'}}, \quad \widehat{q_k^i} = \frac{q_k^i}{\sqrt{\lambda}}.$$

The quantization $P \mapsto \widehat{P}$ of quadratic polynomials is then defined by:

$$\begin{aligned}
\widehat{p_k^i p_l^j} &= \sum_{i'} \sum_{j'} \lambda\, g^{ii'} g^{jj'} \frac{\partial}{\partial q_k^{i'}} \frac{\partial}{\partial q_l^{j'}}, \\
\widehat{p_k^i q_l^j} &= \widehat{q_l^j p_k^i} = q_l^j \sum_{i'} g^{ii'} \frac{\partial}{\partial q_k^{i'}}, \\
\widehat{q_k^i q_l^j} &= \frac{q_k^i q_l^j}{\lambda},
\end{aligned} \tag{5}$$

where a nonsymmetric *choice* is made in the second term.

Let $A \in \mathbb{A}$ be an infinitesimal symplectic transformation. The quantization,

$$\widehat{A} = \widehat{P[A]},$$

is a quadratic differential operator on the Fock space $\mathcal{H}$. A direct calculation shows the action of $\widehat{A}$ on $\mathcal{H}$ is *independent* of the choice of coordinates on H.

The quantization formulas simplify greatly if the coordinates on H are chosen with respect to an orthonormal basis. Most calculations will be done in orthonormal coordinates.

The quantization process is *not* a Lie algebra homomorphism. The following Lemma may be easily verified.

LEMMA 2. *Let F, G be quadratic polynomials. Then,*

$$[\widehat{F}, \widehat{G}] = \widehat{\{F, G\}} + \mathcal{C}(F, G),$$

where the cocycle $\mathcal{C}$, in orthonormal coordinates*, vanishes on all quadratic monomials except:*

$$\begin{aligned}
\mathcal{C}(p_k^i p_l^j, q_k^i q_l^j) &= 1 + \delta^{ij}\delta_{kl}, \\
-\mathcal{C}(q_k^i q_l^j, p_k^i p_l^j) &= 1 + \delta^{ij}\delta_{kl}.
\end{aligned}$$

3. Virasoro operators via quantization

3.1. Virasoro operators. Define the differential operator D by:

$$D = z(\frac{d}{dz})z = z^2\frac{d}{dz} + z.$$

Define the operators $\{L_m\}_{m\geq -1}$ by:

$$L_m = -z^{-1/2}D^{m+1}z^{-1/2}.$$

The operators L_m are seen to have only integer exponents of z by passing $z^{-1/2}$ through D^{m+1}. For example,

$$L_{-1} = -\frac{1}{z},$$

$$L_0 = -\Big(z\frac{d}{dz} + \frac{1}{2}\Big),$$

$$L_1 = -\Big(z^3\frac{d^2}{dz^2} + 3z^2\frac{d}{dz} + \frac{3}{4}z\Big),$$

$$L_2 = -\Big(z^5\frac{d^3}{dz^3} + \frac{15}{2}z^4\frac{d^2}{dz^2} + \frac{45}{4}z^3\frac{d}{dz} + \frac{15}{8}z^2\Big).$$

Hence, the operators L_m act naturally on $\mathbb{H}$.

Lemma 3. *The operators L_m satisfy the following properties:*

(i) $[L_m, L_n] = (m-n)L_{m+n}$,

(ii) $\Omega(L_m f, g) + \Omega(f, L_m g) = 0$.

Proof. After the change of variables $w = 1/z$, we find

$$zDz^{-1} = z^2\frac{d}{dz} = -\frac{d}{dw}.$$

Then, using the definition of L_m,

$$z^{1/2}L_m z^{-1/2} = (-1)^m w\frac{d^{m+1}}{dw^{m+1}}.$$

Property (i) is now easily verified in the variable w.

Property (i) implies the operators $\{L_m\}_{m\geq -1}$ span a Lie algebra generated by L_2 and L_{-1}. Property (ii) then follows from (i) together with elementary verifications in the cases $m = 2, -1$. □

The operators L_m define infinitesimal symplectic transformations by (ii) and can therefore be quantized via the scheme of Section 2.1. In case H has dimension 1, the first few quadratic Hamiltonians (in orthonormal coordinates) are:

$$
\begin{aligned}
P[L_{-1}] =& \frac{1}{2} q_0^2 + \sum_{m=0}^{\infty} q_{m+1} p_m, \\
P[L_0] =& \sum_{m=0}^{\infty} (m + \frac{1}{2})\ q_m p_m, \\
P[L_1] =& \frac{1}{8} p_0^2 + \sum_{m=0}^{\infty} (m + \frac{1}{2})(m + \frac{2}{3})\ q_m p_{m+1}, \\
P[L_2] =& \frac{3}{8} p_0 p_1 + \sum_{m=0}^{\infty} (m + \frac{1}{2})(m + \frac{2}{3})(m + \frac{5}{2})\ q_m p_{m+2}.
\end{aligned}
$$

Let $\widehat{L}'_m$ denote the quantization of the quadratic Hamiltonian $P(L_m)$. The first few quantized operators are:

$$
\begin{aligned}
\widehat{L}'_{-1} &= \frac{1}{2\lambda} q_0^2 + \sum_{m \geq 0} q_{m+1} \partial_m, \\
\widehat{L}'_0 &= \sum_{m \geq 0} (m + \frac{1}{2})\ q_m \partial_m \\
\widehat{L}'_1 &= \frac{1}{8} \lambda \partial_0^2 + \sum_{m \geq 0} (m + \frac{1}{2})(m + \frac{3}{2})\ q_m \partial_{m+1}, \\
\widehat{L}'_2 &= \frac{3}{8} \lambda \partial_0 \partial_1 + \sum_{m \geq 0} (m + \frac{1}{2})(m + \frac{3}{2})(m + \frac{5}{2})\ q_m \partial_{m+2},
\end{aligned}
$$

where $\partial_m = \frac{\partial}{\partial q_m}$.

Define the operators $\widehat{L}_m$ by:

$$\widehat{L}_m = \widehat{L}'_m + \frac{\delta_{m0}}{16}.$$

The operators $\widehat{L}_m$ are obtained from L_m via quantization followed a shift (and hence represent a slight abuse of notation).

Lemma 4. *The operators $\widehat{L}_m$ satisfy the Virasoro bracket:*

$$[\widehat{L}_m, \widehat{L}_n] = (m - n)\widehat{L}_{m+n}.$$

Proof. By Lemmas 2 and 3,

$$[\widehat{L}'_m, \widehat{L}'_n] = (m - n)\widehat{L}'_{m+n}$$

modulo the cocycle term

$$\mathcal{C}(P[L_m], P[L_n]).$$

The cocycle term is seen to vanish in all cases except:

$$\mathcal{C}(P[L_1], P[L_{-1}]) = \frac{1}{8},$$

Here, the cocycle detects the pairing of the monomial p_0^2 in $P[L_1]$ and the monomial q_0^2 in $P[L_{-1}]$. Therefore, the shifted operators $\widehat{L}_m$ satisfy the Virasoro bracket. □

The Lie algebra of holomorphic vector fields on the affine line,

$$\{-z^{m+1}\frac{\partial}{\partial z}\}_{m\geq -1}, \tag{6}$$

satisfies the Virasoro bracket. By Lemma 4, the operators

$$\{\widehat{L}_m\}_{m\geq -1}$$

define a representation of the Lie algebra (6).

3.2. The τ-function of the moduli space of curves. Let q denote the variable set $\{q_k\}_{k\geq 0}$. Let

$$q_+(\psi) = \sum_{k=0}^{\infty}(q_k + \delta_{k1})\psi^k.$$

For genus $g \geq 0$, Define the potential $F_{g+}(q)$ in the dilaton shifted variables q by:

$$F_{g+}(q) = \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\overline{M}_{g,n}} q_+(\psi_1)\ldots q_+(\psi_n),$$

where the unstable moduli spaces are omitted as usual.

Let $\tau_+(\lambda, q)$ be the τ-function of the moduli space of curves in the variables q:

$$\tau_+(\lambda, q) = \exp\left(\sum_{g=0}^{\infty}\lambda^{g-1}F_{g+}(q)\right). \tag{7}$$

We may expand τ_+ in terms of the descendent generating series $F^\bullet_{g+}$ of possibly disconnected curves of genus g,

$$\tau_+(\lambda, q) = \sum_{g\in\mathbb{Z}}\lambda^{g-1}F^\bullet_{g+}(q).$$

Lemma 5. $F^\bullet_{g+}(q) \in \left(-\frac{1}{q_1}\right)^{\frac{1}{24}} q_0^{\max(3-3g,0)} \cdot \mathbb{Q}[q_{\geq 2}, \frac{1}{q_1}][[q_0]].$

PROOF. The factor

$$\left(-\frac{1}{q_1}\right)^{\frac{1}{24}}$$

corresponds to the subfactor

$$\begin{aligned}\exp\left(\lambda^0 \sum_{n=0}^{\infty} \frac{(q_1+1)^n}{n!} \int_{\overline{M}_{1,n}} \psi_1 \cdots \psi_n\right) &= \exp\left(\sum_{n=0}^{\infty} \frac{1}{24} \frac{(q_1+1)^n}{n}\right) \\ &= \exp\left(-\frac{1}{24} \log -q_1\right)\end{aligned}$$

in the definition of τ_+.

Consider the coefficient in $\left(-\frac{1}{q_1}\right)^{-\frac{1}{24}} \cdot F^\bullet_{g+}$ of a fixed monomial in q_0. By dimension constraints, only finitely many topological types of curves may contribute to the coefficient. Therefore, following the dilaton equation analysis of Lemma 32 of Chapter 5, we find

$$F^\bullet_{g+}(q) \in \left(-\frac{1}{q_1}\right)^{\frac{1}{24}} \cdot \mathbb{Q}[q_{\geq 2}, \frac{1}{q_1}][[q_0]].$$

A (possibly disconnected) curve of genus $g \leq 0$ must contain at least $1-g$ components of genus 0. Since $\overline{M}_{0,n}$ has dimension $n-3$, each genus 0 component contributes a q_0^3 factor to $F^\bullet_{g+}$. Hence, $F^\bullet_{g+}$ is divisible by $q_0^{\max(3-3g,0)}$. □

3.3. Virasoro constraints. The Virasoro operators constructed in Section 3.1 annihilate the the τ-function:

$$\widehat{L}_m \ \tau_+(\lambda, q) = 0, \quad \forall m \geq -1. \tag{8}$$

The Virasoro constraints are a fundamental property of the τ-function of the moduli space of curves.

The constraints (8) are equivalent by [**3**] to the KdV equations for the τ-function conjectured by Witten in [**20**]. Witten's conjecture is a consequence of Kontsevich's combinatorial model for the intersection theory of cotangent line classes on $\overline{M}_{g,n}$. Derivations of the combinatorial model can be found in [**14**], [**16**]. The construction of the Virasoro operators via quantization is due to Givental [**9**]

CHAPTER 9

Givental's descendent formula

1. Overview

Givental's formula for the τ-function of a semisimple Frobenius manifold $\mathcal{F}$ is stated here. Two additional structure on $\mathcal{F}$ are required. The first, an R-calibration, was introduced in Chapter 1. The second, a J-calibration, was briefly discussed in Chapter 2. Givental's formula for the τ-function is written in terms of quantized operators associated to the R and J-calibrations of $\mathcal{F}$.

In Chapter 6, Givental's formula for the higher genus potential of $\mathcal{F}$ was stated. The higher genus *descendent* potentials of $\mathcal{F}$ obtained from $\log \tau$ generalize the formulas of Chapter 6.

2. Calibrations and quantization

2.1. Frobenius manifolds. Let $\mathcal{F} = (M, g, A, \mathbf{1})$ be a Frobenius manifold, and let $p \in M$ be a semisimple point. We will study $\mathcal{F}$ near p, either on an open set U is if $\mathcal{F}$ is a complex Frobenius manifold, or on a formal neighborhood, if $\mathcal{F}$ is formal.

Following the notation of Chapter 1, let $\{u^i\}$ be canonical coordinates at p with corresponding vector fields $\{\epsilon_i\}$. Then,

$$\mathbf{1} = \sum_i \epsilon_i.$$

The *normalized* canonical vector fields,

$$\tilde{\epsilon}_i = \sqrt{\Delta^i}\epsilon_i, \tag{9}$$

where

$$\Delta^i = \langle \epsilon_i, \epsilon_i \rangle$$

play a central role. The sign of the square root $\sqrt{\Delta^i}$ is fixed by an R-calibration.

Let $\{t^\mu\}$ be flat coordinates with vector fields $\{\partial_\mu\}$. By convention,

$$\mathbf{1} = \partial_1.$$

The transition matrix between the bases $\{\partial_\mu\}$ and $\{\tilde{\epsilon}_i\}$ of the tangent bundle is Ψ,

$$\partial_\mu = \sum_\mu \Psi^i_\mu \tilde{\epsilon}_i.$$

Together with the additional data of a J-calibration, these structures will be used to define a descendent τ-function for $\mathcal{F}$.

2.2. ***R*-calibrations.** A fundamental solution matrix S in flat coordinates at p defines a basis of ∇_z-flat vector fields:

$$\nabla_z \sum_b S^b_a \partial_b = 0. \tag{10}$$

An R-calibration of $\mathcal{F}$ at $p \in M$, defined in Section 4.5 of Chapter 1, determines the signs of (9) and provides a formal factorization,

$$S = \Psi^{-1} R(z,u) e^{\mathbf{u}/z},$$

where

$$R(z,u) = \sum_{n=0}^{\infty} R_n(u) z^n$$

is matrix series in z. Associated endomorphisms,

$$R_n(u) : TM \to TM,$$

are obtained from the R-calibration by

$$\langle \tilde{\epsilon}_j, R_n \tilde{\epsilon}_i \rangle = (R_n)^j_i.$$

Let $R(z,u)$ also denote the endomorphism series $\sum_{n=0}^{\infty} R_n(u) z^n$.

By definition, R-calibrations are required to satisfy the following properties:

(i) *Identity*: R_0 is the identity operator.
(ii) *Unitarity*: $R(z,u)R^\dagger(-z,u) = 1$.

R-calibrations are most naturally studied in canonical coordinates.

By Lemma 15 of Chapter 1, a canonical endomorphism series $R(z,u)$ is determined at a semisimple point p of a *conformal* Frobenius manifold $\mathcal{F}$.

2.3. ***J*-calibrations.** We will require an additional calibration of $\mathcal{F}$ at p. A J-calibration is a series expansion of a fundamental solution matrix S^b_a of (10) in the variable $1/z$,

$$S^b_a = \sum_{n=0}^{\infty} (J_n)^b_a z^{-n}.$$

An endomorphism series in $1/z$,

$$J(z,t) = \sum_{n=0}^{\infty} J_n z^{-n}, \quad J_n(t) : TM \to TM,$$

is obtained from the J-calibration by

$$J_n \partial_a = \sum_b (J_n)_a^b \partial_b.$$

We require the J-calibration to satisfy two properties:

(i) *Identity*: J_0 is the identity operator.
(ii) *Normalization*: $(J_1)_1^\mu = t^\mu$.

By the identity condition and Lemma 1 of Chapter 1, the J-calibration satisfies the unitary condition:

$$J(z,t)J^\dagger(-z,t) = 1.$$

J-calibrations are most naturally studied in flat coordinates.

The normalization condition links the J-calibration to the choice of flat coordinates. Consider a J-calibration satisfying only the identity property. Truncation of the flatness equations to order $1/z^2$ yields the following differential equations for $(J_1)_1^\mu$:

$$\frac{\partial (J_1)_1^\mu}{\partial t^\nu} = \delta_{\mu\nu}.$$

Hence, there exist constants c^μ for which

$$(J_1)_1^\mu = t^\mu + c^\mu.$$

The normalization condition requires these constant shifts to vanish.

A straightforward series construction from the flatness equation shows the existence of J-calibrations of $\mathcal{F}$ at p. Frobenius manifolds obtained from Gromov-Witten theory carry canonical J-calibrations by Section 5.3 of Chapter 2.

The fundamental matrices S associated to R and J-calibrations may be *unrelated* as the respective z expansions are quite different.

2.4. Completion. Let TM_p be the tangent space of M at p, and let

$$(H, \langle \cdot, \cdot \rangle) = (TM_p, g_p).$$

Since $R(z,p)$ and $J(z,p)$ are series in z and $1/z$, these transformations do *not* act on the symplectic space $\mathbb{H}$. We will require a completion of $\mathbb{H}$ on which both R and J act.

Let $\mathbb{H}^\infty$ denote the vector space of *infinite* series,

$$f(z) = \sum_{k=0}^{\infty} (p_k^1, \ldots, p_k^m) \frac{1}{(-z)^{k+1}} + \sum_{k=0}^{\infty} (q_k^1, \ldots, q_k^m) z^k. \tag{11}$$

The coordinates $\{p_k^\mu, q_k^\mu\}$ are taken with respect to the basis of TM_p defined by the flat vector fields $\{\partial_\mu\}$. Let $\mathbb{H}_-^\infty, \mathbb{H}_+^\infty$ denote the subspaces of negative and positive degree series in z.

Modulo convergence issues, R and J act on $\mathbb{H}^\infty$. There are several technical methods to handle convergence. For J-calibrations of bounded type on Frobenius manifolds defined over a Novikov ring N of an algebraic variety (see Chapter 2), a rather straightforward approach may be taken.

A J-calibration is of *bounded type* if for every d, the matrix coefficients $(J_n)_a^b$ have Novikov parameter Q of degree at least d for sufficiently large n. The condition is satisfied for the canonical J-calibrations of Frobenius manifolds obtained from the Gromov-Witten theory of varieties of bounded type (see Chapter 2).

Define $\bar{\mathbb{H}}$ to be the vector space of series (11) satisfying an additional property: for every d, the coordinates p_k^μ have Novikov coefficients Q of degree at least d for sufficiently large k. For J-calibrations of bounded type on Frobenius manifolds defined over Novikov rings, the space $\bar{\mathbb{H}}$ is a completion of $\mathbb{H}$ on which both R and J act.

For complex Frobenius manifolds, a more subtle approach to the convergence will be explained in Section 3.3.

2.5. Quantization. By the unitary conditions,

$$R(z,p)R^\dagger(-z,p) = J(z,p)J^\dagger(-z,p) = 1,$$

R and J define symplectic transformations on $\bar{\mathbb{H}}$. Since R and J are both endomorphism series starting with the identity, we may express both as exponentials,

$$R = \exp(\log R), \quad J = \exp(\log J).$$

For a symplectic transformation of the form

$$T = \exp(\log T),$$

where $\log(T)$ is necessarily infinitesimally symplectic, the quantization $\widehat{T}$ is *defined* by:

$$\widehat{T} = \exp(\widehat{\log T}).$$

The quantizations $\widehat{R}$ and $\widehat{J}$ are therefore well-defined. The quantization $\widehat{J^{-1}}$ is defined by $\exp(\widehat{\log J^{-1}})$ and is equal to $\widehat{J}^{-1}$.

Since $\log R$ and $\log J$ act on $\bar{\mathbb{H}}$ rather than $\mathbb{H}$, the associated quadratic Hamiltonians need not be quadratic *polynomials* in the variables $\{p_k^i, q_k^i\}$. A single variable may occur infinitely often.

2.6. Change of basis. Darboux coordinates on $\bar{\mathbb{H}}$ are also determined by the canonical structure:

$$f(z) = \sum_{k=0}^{\infty}(p_k^1, \ldots, p_k^m)\frac{1}{(-z)^{k+1}} + \sum_{k=0}^{\infty}(q_k^1, \ldots, q_k^m)z^k,$$

where the coordinates $\{p_k^i, q_k^i\}$ are taken *with respect to the basis of* TM_p *defined by the vector fields* $\{\tilde{\epsilon}_i\}$.

The quantization $\widehat{R}_c$ of R with respect to the canonical frame $\{\tilde{\epsilon}_i\}$ is a quadratic differential operator in the variables $\mathbf{q}$. We will primarily use the quantization $\widehat{R}_c$ with respect to the canonical frame rather than the quantization $\widehat{R}$ with respect to the flat frame.

Define the action of the operator Ψ^{-1} on the Fock space of functions in the variables $\mathbf{q}$ by the following rule:

$$\Psi^{-1}\Big(\gamma(\lambda, \mathbf{q})\Big) = \gamma(\lambda, \Psi\mathbf{q}),$$

where Ψ on the right acts as the linear change of basis transformation,

$$(\Psi q)_k^i = \sum_{\mu} \Psi_\mu^i q_k^\mu.$$

Similarly, let Ψ act on the Fock space by

$$\Psi\Big(\gamma(\lambda, \mathbf{q})\Big) = \gamma(\lambda, \Psi^{-1}\mathbf{q}), \quad (\Psi^{-1}\mathbf{q})_k^\mu = \sum_i (\Psi^{-1})_i^\mu q_k^i.$$

The operators $\Psi^{\pm 1}$ on Fock space simply transform functions by the change of basis relating the frames $\{\partial_\mu\}$ and $\{\tilde{\epsilon}_i\}$. The change of basis formula,

$$\widehat{R}_c = \Psi \widehat{R} \Psi^{-1},$$

relating the quantizations of R is easily verified.

2.7. ***M*-dependence.** For points $p' \in M$ near p, the flat frame $\{\partial_\mu\}$ determines a isomorphism

$$(TM_{p'}, g_{p'}) \overset{\sim}{=} (TM_p, g_p). \tag{12}$$

The operators $R(z,u)$ and $J(z,t)$ may be viewed as M-dependent symplectic transformations of $\bar{\mathbb{H}}$ yielding M-dependent quantizations $\widehat{R}$ and $\widehat{J}$.

The canonical frame $\{\tilde{\epsilon}_i\}$ also determines an isomorphism (12). The operator $\widehat{R}_c$ will be viewed as the M-dependent quantization of R with respect to the canonical frame.

As before, the change of basis operator $\Psi^{\pm 1}$ relates the two quantizations.

2.8. Two formulas. An M-dependent symplectic transformation of $\bar{\mathbb{H}}$ of the form

$$T = \exp(\log T)$$

is an element of the *symplectic loop group* if

$$T(z,t) = \sum_{k \in \mathbb{Z}} T_k(t) z^k,$$

where each $\xi_k(t) : H \to H$ is a linear transformation. The calibrations R and J are elements of the symplectic loop group.

LEMMA 6. *If T is an element of the symplectic loop group, then $T^{-1}dT$ is an infinitesimal symplectic transformation.*

PROOF. The differential d in dT is taken on the manifold M,

$$dT = \sum_i \frac{\partial T}{\partial t^\mu} dt^\mu.$$

Let $f_1, f_2 \in \bar{\mathbb{H}}$. Since Ω and the vectors f_i have no t dependence,

$$d\left(\Omega(Tf_1, Tf_2)\right) = d\left(\Omega(f_1, f_2)\right) = 0.$$

Since

$$\begin{aligned} d\left(\Omega(Tf_1, Tf_2)\right) &= \Omega(dTf_1, Tf_2) + \Omega(Tf_1, dTf_2) \\ &= \Omega(T^{-1}dTf_1, f_2) + \Omega(f_1, T^{-1}dTf_2), \end{aligned}$$

we conclude $T^{-1}dT$ is a (1-form valued) infinitesimal symplectic transformation. $\square$

LEMMA 7. *If T is an element of the symplectic loop group for which $T_k = 0$ for* either *all $k \geq 0$ or all $k < 0$, then*

$$\widehat{T^{-1}dT} = \widehat{T}^{-1} d\widehat{T}.$$

PROOF. Since $T^{-1}dT$ is an infinitesimal symplectic transformation by Lemma 6, the quantization $\widehat{T^{-1}dT}$ is well-defined. In fact, for a vector v,

$$\iota_v\, \widehat{T^{-1}dT} = \lim_{\delta \to 0} \frac{(T^{-1}(t)T(t+\delta v))\widehat{\ } - (T^{-1}(t)T(t))\widehat{\ }}{\delta}.$$

The Lemma follows from the definition of $d\widehat{T}$ if

$$(T^{-1}(t)T(t+\delta v))\widehat{\ } = \widehat{T^{-1}(t)}\widehat{T^{-1}(t+\delta v)}. \tag{13}$$

We now prove (13). For notational convenience, let

$$A_1 = \log(T^{-1}(t)), \quad A_2 = \log(T(t+\delta v)).$$

The vanishing assumptions on T and Lemma 2 imply

$$[\widehat{A_1}, \widehat{A_2}] = \widehat{[A_1, A_2]} \tag{14}$$

since all the cocycles vanish. By induction, the corresponding equation holds for all iterated commutation expressions. For example,

$$[\widehat{A_1}, [\widehat{A_1}, \widehat{A_2}]] = [A_1, [A_1, A_2]]\widehat{\ }.$$

By the Baker-Campbell-Hausdorff formula,

$$\log\big(\exp A_1 \exp A_2\big) = A_1 + A_2 + \frac{1}{2}[A_1, A_2] + \ldots,$$

where the omitted terms are iterated commutator expressions. By the iterates of equation (14),

$$\Big(\log\big(\exp A_1 \exp A_2\big)\Big)\widehat{\ } = \widehat{A_1} + \widehat{A_2} + \frac{1}{2}[\widehat{A_1}, \widehat{A_2}] + \ldots,$$

By the definition of quantization and the Baker-Campbell-Hausdorff formula,

$$\begin{aligned} \widehat{T^{-1}(t)T(t+\delta v)} &= \exp\left(\Big(\log\big(\exp A_1 \exp A_2\big)\Big)\widehat{\ }\right) \\ &= \exp\widehat{A_1}\exp\widehat{A_2} \\ &= \widehat{T^{-1}(t)}\widehat{T^{-1}(t+\delta v)}, \end{aligned}$$

completing the proof. □

Let B be an endomorphism of H which defines an infinitesimal symplectic transformation of $\bar{\mathbb{H}}$ (with no z dependence).

Lemma 8. *If T is an element of the symplectic loop group for which $T_k = 0$ for* either *all $k \geq 0$ or all $k < 0$, then*

$$\widehat{T^{-1}BT} = \widehat{T}^{-1}\widehat{B}\widehat{T}.$$

Proof. The left side is well-defined since $T^{-1}BT$ is an infinitesimal symplectic transformation. Let $A = \log(T)$. Then,

$$T^{-1}BT = -[A, B] + \frac{1}{2}[A,[A,B]] - \frac{1}{6}[A,[A,[A,B]]] + \ldots \tag{15}$$

The vanishing assumptions on T and Lemma 2 imply

$$[\widehat{A},\widehat{B}] = \widehat{[A,B]} \tag{16}$$

since all the cocycles vanish. As before, the corresponding equation holds for all iterated commutation expressions. We conclude,

$$\begin{aligned}
\widehat{T^{-1}BT} &= \Big(B - [A,B] + \frac{1}{2}[A,[A,B]] - \frac{1}{6}[A,[A,[A,B]]] + \ldots\Big)^{\widehat{}} \\
&= \Big(\widehat{B} - [\widehat{A},\widehat{B}] + \frac{1}{2}[\widehat{A},[\widehat{A},\widehat{B}]] - \frac{1}{6}[\widehat{A},[\widehat{A},[\widehat{A},\widehat{B}]]] + \ldots\Big) \\
&= \widehat{T}^{-1}\widehat{B}\widehat{T},
\end{aligned}$$

completing the proof. □

3. The τ-function of a semisimple Frobenius manifold

3.1. Variables. Let $\mathcal{F}$ be a complex or formal Frobenius manifold of dimension m with R and J-calibrations. Let p be a semisimple point of $\mathcal{F}$. We associate descendent variables,

$$\mathbf{t} = \{\mathbf{t}_k^{\mu}\}_{k\geq 0},$$

to the flat coordinates $\{t^{\mu}\}$. We associate Fock space variables,

$$\mathbf{q} = \{q_k^i\}_{k\geq 0},$$

to the canonical coordinates $\{u^i\}$. Let

$$q^i = \{q_k^i\}_{k\geq 0}$$

denote the subset of variables with fixed upper index. Similarly, let

$$q_k = \{q_k^i\}^{1\leq i\leq m}$$

denote the subset with fixed upper index.

The τ-function of $\mathcal{F}$ at p is written in terms of the variables $\mathbf{t}$. However, the Fock space variables $\mathbf{q}$ will play a central role in the definition.

3.2. The τ-function. The τ-function of $\mathcal{F}$ at a point p' near p is *defined* by:

$$\tau_{p'}^{\mathcal{F}}(\lambda,\mathbf{t}) = \left[e^{\frac{1}{2}\int_p^{p'}\sum (R_1)_i^i du^i}\, \widehat{J}^{-1}\Psi^{-1}\widehat{R}_c\; e^{\widehat{\mathbf{u}/z}} \prod_{i=1}^{m} \tau_+(\lambda, q^i)\right]_{\mathbf{q}_+=\mathbf{t}} . \tag{17}$$

We discuss the terms in the formula from right to left. The factors of the product

$$\prod_{i=1}^{m} \tau_+(\lambda, q^i)$$

are τ-functions of the moduli space of curves (7) in the variables q^i. The product is interpreted as an element of the Fock space of functions in the variables $\mathbf{q}$ with respect to the canonical frame $\{\tilde{\epsilon}_i\}$.

The symbol $\mathbf{u}$ denotes the diagonal matrix with entries u^i. By the string equation,

$$\widehat{\frac{u^i}{z}}\ \tau_+(\lambda, q^i) = -\widehat{L}_{-1}\ \tau_+(\lambda, q^i) = 0,$$

see Sections 3.1-3.3. Therefore,

$$e^{\widehat{\mathbf{u}/z}} \prod_{i=1}^{m} \tau_+(\lambda, q^i) = \prod_{i=1}^{m} \tau_+(\lambda, q^i).$$

The operator $e^{\widehat{\mathbf{u}/z}}$ is included here only for aesthetic compatibility with Theorem 1 of Chapter 1.

The operators $\widehat{R}_c$, Ψ^{-1}, and $\widehat{J}^{-1}$ at p' are defined in Section 2. Since the change of basis operator Ψ^{-1} occurs, the function

$$\Psi^{-1}\widehat{R}_c\ e^{\widehat{\mathbf{u}/z}} \prod_{i=1}^{m} \tau_+(\lambda, q^i)$$

is interpreted as an element of the Fock space of functions in the variables

$$\mathbf{q} = \{\mathbf{q}_k^\mu\}$$

with respect to the flat frame $\{\partial_\mu\}$. The prefactor

$$e^{\frac{1}{2}\int_p^{p'} \sum (R_1)_i^i du^i}$$

is a constant independent of $\mathbf{q}$ (but dependent on the point p').

The final change of variables $\mathbf{q}_+ = \mathbf{t}$ is defined by the equations:

$$\begin{aligned} q_1^1 + 1 &= \mathbf{t}_1^1, \\ q_{k\neq 1}^{\mu\neq 1} &= \mathbf{t}_k^\mu. \end{aligned}$$

In Section 3.3, we will prove $\tau_{p'}^{\mathcal{F}}(\lambda, \mathbf{t})$ is a power series in a set of *shifted* descendent variables $\mathbf{t}$ depending upon the point p'. The τ-function will be proven to be *independent* of p' in Sections 3.4 and 4.2.

The τ-function will yield *descendent* potentials for $\mathcal{F}$ generalizing the potentials for primary fields defined in Chapter 6.

3.3. Convergence. Our first goal is to prove the function $\tau_{p'}^{\mathcal{F}}(\lambda, \mathbf{t})$ is well-defined. The matter is subtle since R and J^{-1} have respectively positive and negative power expansions in z. At the end of the Section, a more explicit formula for $\tau_{p'}^{\mathcal{F}}$ will be presented using alternate descriptions of the operators $\widehat{R}_c$ and $\widehat{J}^{-1}$.

Quantized operators act on Fock spaces of functions $\gamma(\lambda, \mathbf{q})$ in the variables $\mathbf{q}$. We define the Fock space $\mathbb{F}_c$ to be the set of functions,

$$\gamma(\lambda, \mathbf{q}) = \alpha(\mathbf{q}) \sum_{g \in \mathbb{Z}} \lambda^{g-1} \gamma_g(\mathbf{q}),$$

where

$$\alpha(\mathbf{q}) \in \mathbb{C}[\ (-q_1)^{-\frac{1}{24}}\ ]$$

and

$$\gamma_g \in q_0^{\max(3-3g,0)} \cdot \mathbb{C}[q_{\geq 1}, \frac{1}{q_1}][[q_0]].$$

Here,

$$\left(-\frac{1}{q_1}\right)^{\frac{1}{24}} = \left\{ \left(-\frac{1}{q_1^1}\right)^{\frac{1}{24}}, \ldots, \left(-\frac{1}{q_1^m}\right)^{\frac{1}{24}} \right\},$$

The Fock space $\mathbb{F}_c$ is closed under multiplication. Therefore, by Lemma 5,

$$\prod_{i=1}^{m} \tau_+(\lambda, q^i)$$

lies in the Fock space.

Lemma 9. *The operator $\widehat{R}_c$ preserves the Fock space $\mathbb{F}_c$.*

Proof. The operator $\widehat{R}_c$ is defined as $\exp(\widehat{\log R_c})$. We obtain the following formula from the definition of quantization in the canonical frame:

$$\widehat{\log R_c} = -\sum_{i,j,k,l} \big((\log R)_{k-l}\big)^i_j q^j_l \partial_{q^i_k} + \frac{\lambda}{2} \sum_{i,j,k,l} (-1)^l \big((\log R)_{k+l+1}\big)^i_j \partial_{q^i_k} \partial_{q^j_l}.$$

Since the z expansion of $\log R$ has only positive coefficients, the indicies of the first sum must satisfy $k > l$.

An elementary verification using the above formula and the definition of the Fock space shows $\exp(\widehat{\log R_c})$ acts on $\mathbb{F}_c$. $\square$

Lemma 10. *Let $\gamma(\lambda, \mathbf{q})$ be an element in the Fock space $\mathbb{F}_c$. Then,*

$$\left[e^{\frac{\lambda}{2} \sum E^{ij}_{kl} \frac{\partial}{\partial Q^i_k} \frac{\partial}{\partial Q^j_l}} \gamma(\lambda, \mathbf{Q}) \right]_{\mathbf{Q}=R^{-1}\mathbf{q}} \in \mathbb{F}_c.$$

Proof. The functions E^{ij}_{kl} are defined in Section 4 of Chapter 6,

$$\sum_{k,l \geq 0}^{\infty} E^{ij}_{kl} (-w)^k (-z)^l = \frac{1}{w+z} \Big(\sum_m R^m_i(w) R^m_j(z) - \delta_{ij} \Big). \tag{18}$$

The variables $\mathbf{Q}$ are determined by

$$Q^i_k = \sum_{l=0}^{k} \sum_j (R^{-1}_{k-l})^i_j q^j_l.$$

The function $e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}}\gamma(\lambda,\mathbf{Q})$ is easily seen to be an element of the Fock space $\mathbb{F}_c$ in the variables $\mathbf{Q}$. Via the expansion

$$\begin{aligned} -\frac{1}{Q^i_1} &= -\frac{1}{q^i_1 + \sum_j (R^{-1}_1)^i_j q^j_0} \\ &= \sum_{k\geq 0} -\left(\frac{1}{q^i_1}\right)\left(-\frac{\sum_j (R^{-1}_1)^i_j q^j_0}{q^i_1}\right)^k, \end{aligned}$$

the change of variables $\mathbf{Q} = R^{-1}\mathbf{q}$ preserves the Fock space. □

PROPOSITION 1. *Let $\gamma(\lambda,\mathbf{q})$ be an element in the Fock space $\mathbb{F}_c$. Then,*

$$(\widehat{R}_c\gamma)(\lambda,\mathbf{q}) = \left[e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}}\gamma(\lambda,\mathbf{Q})\right]_{\mathbf{Q}=R^{-1}\mathbf{q}}.$$

PROOF. Let $R[s](z) = \exp(s\log R(z))$. We will prove a more general relation:

$$(19)\qquad \left(\widehat{R[s]}_c\gamma\right)(\lambda,\mathbf{q}) = \left[e^{\frac{\lambda}{2}\sum E^{ij}_{kl}[s]\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}}\gamma(\lambda,\mathbf{Q})\right]_{\mathbf{Q}=R[s]^{-1}\mathbf{q}}.$$

Here, $E^{ij}_{kl}[s]$ is defined by (18) with $R[s]$ instead of R on the right. The function E^{ij}_{kl} is *polynomial* in s.

Certainly (19) holds when $s=0$. We will show the two sides of (19) satisfy the same differential equation with respect to s. The s dependence of a fixed monomial in the variables λ and q_0 occurring on either side of (19) is polynomial. Hence, we can conclude equation (19) holds for all s. The Proposition is obtained when $s=1$.

Let $\star_r$ denote the right side of (19). After differentiation, we find,

$$(20)\quad \frac{d}{ds}(\star_r) = \sum\left(\frac{dR[s]^{-1}_{k-l}}{ds}\right)^i_j q^j_l \partial_{Q^i_k}(\star_r) + \frac{\lambda}{2}\sum\frac{dE^{ij}_{kl}[s]}{ds}\partial_{Q^i_k}\partial_{Q^j_l}(\star_r).$$

Both sums are over all the indices i,j,k,l. The derivatives in (20) are to be taken *before* the change of variables to $\mathbf{q}$.

Consider the first term on the right of (20). Using the differentiation

$$\frac{dR[s]^{-1}}{ds} = -R[s]^{-1}\log R$$

and the chain rule

$$\partial_{Q^i_k} = \sum_{i',k'} (R[s]_{k'-k})^{i'}_i \partial_{q^{i'}_{k'}},$$

the first term of (20) may be rewritten as

$$-\sum \left((\log R)_{k-l}\right)^i_j q^j_l \partial_{q^i_k}(\star_r). \tag{21}$$

The analysis of the second term of (20) is more complicated. We write the derivative of (18) in matrix form,

$$\sum \frac{dE^{ij}_{kl}[s]}{ds}(-w)^k(-z)^l = \frac{1}{w+z}\frac{d}{ds}\left(R^t[s](w)R[s](z)\right)^i_j \tag{22}$$

Expansion of the matrix derivative in (22) yields:

$$\left(R^t[s](w)\big(\frac{-\log R(-w)+\log R(z)}{w+z}\big)R[s](z)\right)^i_j.$$

Here, the unitary condition $\log R^t(w) = -\log R(-w)$ is used. Together with chain rule and algebraic manipulation, we write the second term (20) as

$$\frac{\lambda}{2}\sum(-1)^l\left((\log R)_{k+l+1}\right)^i_j \partial_{q^i_k}\partial_{q^j_l}(\star_r).$$

Hence, the derivative $\frac{d\star_r}{ds}$ may be written as:

$$-\sum \left((\log R)_{k-l}\right)^i_j q^j_l \partial_{q^i_k}(\star_r) + \frac{\lambda}{2}\sum(-1)^l\left((\log R)_{k+l+1}\right)^i_j \partial_{q^i_k}\partial_{q^j_l}(\star_r),$$

where the sums are over i, j, k, and l (with $k > l$ in the first sum).

Let $\star_l$ denote the left side of (19). The differential equation,

$$\frac{d}{ds}(\star_l) = \widehat{\log R}_c(\star_l)$$

is immediately obtained. The quantization $\widehat{\log R_c}$ has been explicitly computed in Lemma 9. After comparison, we conclude the functions $\star_l$ and $\star_r$ satisfy the same differential equation in s. □

The next step is to apply the change of basis operator Ψ^{-1} to the function

$$\widehat{R}_c\, e^{\frac{\hat{\mathbf{u}}}{z}} \prod_{i=1}^m \tau_+(\lambda, q^i).$$

The operator Ψ^{-1} takes $\mathbb{F}_c$ to a corresponding Fock space $\mathbb{F}_f$ for flat coordinates.

We define the Fock space $\mathbb{F}_f$ to be the set of functions,

$$\gamma(\lambda, \mathbf{q}) = \alpha(\mathbf{q})\sum_{g\in\mathbb{Z}} \lambda^{g-1}\gamma_g(\mathbf{q}),$$

where

$$\alpha(\mathbf{q}) \in \mathbb{C}[\ \left(-\frac{1}{q_1^1}\right)^{\frac{1}{24}}\ ]$$

and

$$\gamma_g \in q_0^{\max(3-3g,0)} \cdot \mathbb{C}[q_{\geq 2}, q_1^1, \frac{1}{q_1^1}][[q_0, q_1^{\mu\neq 1}]].$$

LEMMA 11. *The operator Ψ^{-1} takes $\mathbb{F}_c$ to $\mathbb{F}_f$ modulo $\mathbb{C}^*$.*

PROOF. Recall the action of the change of basis operator defined in Section 2.6:

$$\Psi^{-1}\Big(\gamma(\lambda,\mathbf{q})\Big) = \gamma(\lambda, \Psi\mathbf{q}).$$

Here, Ψ on the right acts as the linear change of basis transformation,

$$(\Psi\mathbf{q})_k^i = \sum_i \Psi_\mu^i q_k^\mu.$$

The change of basis preserves the lower subscript of the variables.

Following the notation of Chapter 1, the unit for the $*$-product in the flat frame is

$$\partial_1 = \sum_{i=1}^m \frac{1}{\sqrt{\Delta_i}}\tilde{\epsilon}_i.$$

Hence, by definition,

$$\Psi_1^i = \frac{1}{\sqrt{\Delta_i}}.$$

The expansion,

$$(23)\qquad \begin{aligned} -\frac{1}{(\Psi q)_1^i} &= -\frac{1}{\frac{1}{\sqrt{\Delta^i}}q_1^1 + \sum_{\mu\neq 1}\Psi_\mu^i q_1^\mu} \\ &= \sqrt{\Delta^i}\sum_{k\geq 0}\left(-\frac{1}{q_1^1}\right)\left(-\frac{\sqrt{\Delta^i}\sum_{\mu\neq 1}\Psi_\mu^i q_1^\mu}{q_1^1}\right)^k, \end{aligned}$$

then proves the Lemma.

The expansion of $\left(-\frac{1}{(\Psi q)_1^i}\right)^{\frac{1}{24}}$ obtained from (23) requires the root extraction

$$(\sqrt{\Delta^i})^{\frac{1}{24}}.$$

Therefore, the operator Ψ introduces a single scalar ambiguity by a 24^{th} root of unity. $\square$

The final step is to apply the operator $\widehat{J}^{-1}$ to the function

$$\Psi^{-1}\widehat{R}_c\, e^{\frac{\hat{\mathbf{u}}}{z}}\prod_{i=1}^m \tau_+(\lambda, q^i).$$

We will introduce new Fock spaces for the action of $\widehat{J}^{-1}$.

We define the Fock space $\mathbb{F}_{\mathbf{t}}$ to be the set of functions,

$$f(\lambda, \mathbf{t}) = \exp(\frac{\kappa}{\lambda}) \sum_{g \in \mathbb{Z}} \lambda^{g-1} f_g(\mathbf{t}),$$

where $\kappa \in \mathbb{C}$ is a constant and $f_g \in \mathbb{C}[[\mathbf{t}]]$. A canonical embedding,

$$\mathbb{F}_f/\mathbb{C}^* \subset \mathbb{F}_{\mathbf{t}}/\mathbb{C}^*,$$

is defined by the change of variables,

$$\begin{aligned} q_1^1 + 1 &= \mathbf{t}_1^1, \\ q_{k\neq 1}^{\mu\neq 1} &= \mathbf{t}_k^\mu, \end{aligned} \tag{24}$$

and the series expansion,

$$\frac{1}{q_1^1} = \frac{1}{\mathbf{t}_1^1 - 1} = -\sum_{k\geq 0} (\mathbf{t}_1^1)^k.$$

The root extraction $(-1)^{\frac{1}{24}}$ again introduces a scalar ambiguity. For the above embedding, $\kappa = 0$.

Let $t^\mu(p')$ be the flat coordinates of the point $p' \in M$. For each value of the parameter s, we define a variable set $\mathbf{t}[s]$ by:

$$\begin{aligned} \mathbf{t}[s]_0^\mu &= \mathbf{t}_0^\mu - s \cdot t^\mu(p') \\ \mathbf{t}[s]_{k\neq 0}^\mu &= \mathbf{t}_k^\mu. \end{aligned} \tag{25}$$

The variables $\mathbf{q}$ and $\mathbf{t}[s]$ are related by the definitions (24) and (25):

$$\begin{aligned} q_0^\mu &= \mathbf{t}[s]_0^\mu + s \cdot t^\mu(p'), \\ q_1^1 &= \mathbf{t}[s]_1^1 - 1, \\ q_k^\mu &= \mathbf{t}[s]_k^\mu \ \text{ otherwise.} \end{aligned} \tag{26}$$

We define the Fock space $\mathbb{F}_{\mathbf{t}[s]}$ in the variables $\mathbf{t}[s]$ to be the set of functions,

$$f(\lambda, \mathbf{t}[s]) = \exp(\frac{\kappa}{\lambda}) \sum_{g \in \mathbb{Z}} \lambda^{g-1} f_g(\mathbf{t}[s]),$$

where $\kappa \in \mathbb{C}$ is a constant and $f_g \in \mathbb{C}[[\mathbf{t}[s]]]$. We will often denote $\mathbf{t}[1]$ by $\mathbf{t}_{p'}$.

Lemma 12. *The operator $\widehat{J}^{-1}$ at p' maps $\mathbb{F}_{\mathbf{t}}$ to $\mathbb{F}_{\mathbf{t}_{p'}}$.*

Proof. We obtain the following formula for $-\widehat{\log J}$ from the definition of quantization in the canonical frame:

$$\sum_{i,j,k,l} (\log J_{l-k})_j^i \, q_l^j \frac{\partial}{\partial q_k^i} + \frac{1}{2\lambda} \sum_{i,j,k,l} \sum_{i'} (-1)^k g_{ii'} \big((\log J)_{k+l+1} \big)_j^{i'} q_k^i q_l^j.$$

Since the z expansion of $\log J$ has only positive coefficients, the indicies of the first sum must satisfy $l > k$.

We can easily write $-\widehat{\log J}$ as a differential operator in the descendent variables $\mathbf{t}$:

$$\sum_{i,j,k,l} (\log J_{l-k})^i_j \, \mathbf{t}^j_l \frac{\partial}{\partial \mathbf{t}^i_k} - \sum_i t^i(p') \frac{\partial}{\partial \mathbf{t}^i_0}$$
$$+ \frac{1}{2\lambda} \sum_{i,j,k,l} \sum_{i'} (-1)^k g_{ii'} \left((\log J)_{k+l+1} \right)^{i'}_j (\mathbf{t}^i_k - \delta_{i1}\delta_{k1})(\mathbf{t}^j_l - \delta_{j1}\delta_{l1}).$$

The normalization condition, $(J_1)^\mu_1 = t^\mu$, is used in the derivation.

We now describe the action of $\widehat{J}^{-1}$ on $\mathbb{F}_{\mathbf{t}}$. The operator $\widehat{J}^{-1}$ is defined by $\exp(-\widehat{\log J})$. After persistent application of the Baker-Campbell-Hausdorff formula, we may write $\widehat{J}^{-1}$ as:

(27)
$$\exp\left(\frac{\kappa(\mathbf{t})}{\lambda}\right) \exp\left(\sum_{i,j,k,l} (\log J_{l-k})^i_j \, \mathbf{t}^j_l \frac{\partial}{\partial \mathbf{t}^i_k} \right) \exp\left(-\sum_i t^i(p') \frac{\partial}{\partial \mathbf{t}^i_0} \right).$$

Here, $\kappa(\mathbf{t})$ is a series of order at most 2 in the variables $\mathbf{t}$. Let

$$f(\lambda, \mathbf{t}) \in \mathbb{F}_{\mathbf{t}}.$$

The first term on the right of (27) is *defined* to act by shifting the $\mathbf{t}_0$ variables:

$$\exp\left(-\sum_i t^i(p') \frac{\partial}{\partial \mathbf{t}^i_0} \right) \; f(\lambda, \mathbf{t}) = f(\lambda, \mathbf{t}_{p'}) \in \mathbb{F}_{\mathbf{t}_{p'}}.$$

Since $l > 0$ in the sum, the second term of (27) certainly acts on $\mathbb{F}_{\mathbf{t}_{p'}}$. The final term of (27) acts by expanding the exponential (except for the constant). $\square$

We define the functions W^{ij}_{kl} by the following expansion coefficients,

$$\frac{\sum_{i',m} g_{ii'} (J^\dagger(w))^{i'}_m (J(z))^m_j - g_{ij}}{w^{-1} + z^{-1}} = \sum_{k,l=0}^{\infty} W^{ij}_{kl} w^{-k} z^{-l}. \tag{28}$$

The denominator on the left divides the numerator by the unitary property of the J-calibration. The symmetry,

$$W^{ij}_{kl} = W^{ji}_{lk},$$

follows from the definition.

Define an associated quadratic form,

$$W(\mathbf{q}) = \sum W^{ij}_{kl} q^i_k q^j_l, \tag{29}$$

and define

$$W_{p'}(\mathbf{t}_{p'}) = W(\mathbf{q})$$

via the relation (26) between the variables $\mathbf{q}$ and $\mathbf{t}_{p'}$.

LEMMA 13. *Let $f(\lambda, \mathbf{t})$ be an element in the Fock space $\mathbb{F}_{\mathbf{t}}$. Then,*

$$e^{\frac{W_{p'}(\mathbf{t}_{p'})}{2\lambda}} f(\lambda, \lfloor J\mathbf{t}_{p'} \rfloor) \in \mathbb{F}_{\mathbf{t}_{p'}},$$

where the substitution $\mathbf{t} = \lfloor J\mathbf{t}_{p'} \rfloor$ is defined by

$$\mathbf{t}^i_k = \sum_{l=k}^{\infty} \sum_j (J_{l-k})^i_j \mathbf{t}^j_{l,p'}.$$

PROOF. The Lemma follows from the definitions of the Fock spaces and the substitution rule. □

The action of $\widehat{J}^{-1}$ on $\mathbb{F}_{\mathbf{t}}$ is determined by the following Proposition.

PROPOSITION 2. *Let $f(\lambda, \mathbf{t})$ be an element in the Fock space $\mathbb{F}_{\mathbf{t}}$. Then,*

$$(\widehat{J}^{-1}\gamma)(\lambda, \mathbf{t}) = e^{\frac{W_{p'}(\mathbf{t}_{p'})}{2\lambda}} f(\lambda, \lfloor J\mathbf{t}_{p'} \rfloor).$$

PROOF. For simplicity, we present the proof in case $f(\lambda, \mathbf{t})$ is obtained from an element

$$\gamma(\lambda, \mathbf{q}) \in \mathbb{F}_f.$$

We will use the variables $\mathbf{q}$ throughout the proof. The minor modifications required in the general case are left to the reader.

We follow the strategy of the proof of Proposition 1. Let

$$J[s](z) = \exp(s \log J(z)).$$

We will prove a more general relation:

$$\left(\widehat{J[s]}^{-1}\gamma\right)(\lambda, \mathbf{q}) = e^{\frac{W[s](\mathbf{q})}{2\lambda}} \Big[\gamma(\lambda, \mathbf{Q})\Big]_{\mathbf{Q}=\lfloor J[s]\mathbf{q}\rfloor}, \tag{30}$$

$$Q^i_k = \sum_{l=k}^{\infty} \sum_j (J_{l-k})^i_j q^j_l.$$

Here, $W[s]$ is defined by equations (18) and (29) with $J[s]$ instead of J on the right. The function W^{ij}_{kl} is *polynomial* in s.

Both sides of (30) are elements of $\mathbb{F}_{\mathbf{t}[s]}$ via the variable relations (26). Certainly (30) holds when $s = 0$. We will show the two sides of (30) satisfy the same differential equation with respect to s. The s dependence of a fixed monomial in the variables λ and $\mathbf{t}[s]$ occurring on either side of (30) is polynomial. Hence, we can conclude equation (30) holds for all s. The Proposition is obtained when $s = 1$.

Let $\star_r$ denote the right side of (30). After differentiation, we find,

$$\begin{aligned}\frac{d}{ds}(\star_r) &= e^{\frac{W[s](\mathbf{q})}{2\lambda}}\Big[\sum_{i,j,k,l}\left(\frac{dJ[s]_{l-k}}{ds}\right)^i_j q^j_l \frac{\partial}{\partial Q^i_k}\gamma(\lambda,\mathbf{Q})\Big]_{\mathbf{Q}=\lfloor J[s]\mathbf{q}\rfloor} \\ &\quad +\Big(\frac{1}{2\lambda}\frac{\partial}{\partial s}W[s](\mathbf{q})\Big)(\star_r).\end{aligned}$$

Using the differentiation

$$\frac{dJ[s]}{ds} = J[s](\log J)$$

and the chain rule

$$\partial_{Q^i_k} = \sum_{i',k'}(J[s]^{-1}_{k-k'})^{i'}_i \partial_{q^{i'}_{k'}},$$

the first term of $\frac{d}{ds}(\star_r)$ may be rewritten as

$$e^{\frac{W[s](\mathbf{q})}{2\lambda}}\sum_{i,j,k,l}(\log J_{l-k})^i_j\, q^j_l \frac{\partial}{\partial q^i_k}\ \gamma(\lambda,\lfloor J[s]\mathbf{q}\rfloor). \tag{31}$$

We now analyze the second term of $\frac{d}{ds}(\star_r)$. Differentiation of the defining equation (29) of $W[s](\mathbf{q})$ with respect to s and the unitary condition,

$$\log J^\dagger(w) = -\log J(-w),$$

together yield the following result:

$$\begin{aligned}&\sum_{k,l}\left(\frac{\partial}{\partial s}W^{ij}_{kl}[s]\right)w^{-k}z^{-l} = \\ &\qquad\sum_{i'} g_{ii'}\Big(J^\dagger[s](w)\left(\sum(-1)^k(\log J)_{k+l+1}w^{-k}z^{-l}\right)J[s](z)\Big)^{i'}_j.\end{aligned} \tag{32}$$

We then find

$$\begin{aligned}\frac{\partial}{\partial s}W[s](\mathbf{q}) &= \sum_{i,j,k,l}\frac{\partial}{\partial s}W^{ij}_{kl}[s]\ q^i_k q^j_l \\ &= \sum_{i,j}(-1)^k\sum_{k,l,a,b}\sum_{i'} g_{ii'}\left(J^\dagger[s]_a(\log J)_{k+l+1}J[s]_b\right)^{i'}_j q^i_{a+k}q^j_{b+l} \\ &= B(\lfloor J[s]\mathbf{q}\rfloor),\end{aligned}$$

where the bilinear form $B(\mathbf{q})$ is defined to be

$$B(\mathbf{q}) = \sum_{i,j,k,l}\sum_{i'}(-1)^k g_{ii'}\big((\log J)_{k+l+1}\big)^{i'}_j q^i_k q^j_l.$$

Since $W[0](\mathbf{q}) = 0$,

$$W[s](\mathbf{q}) = \int_0^s \frac{\partial}{\partial s} W[s](\mathbf{q}) ds = \int_0^s B(\lfloor J[s]\mathbf{q} \rfloor) ds.$$

By differentiating under the integral, we find

$$\begin{aligned}
\sum_{i,j,k,l} (\log J_{l-k})^i_j \, q^j_l \frac{\partial W[s](\mathbf{q})}{\partial q^i_k} &= \int_0^s \sum_{i,j,k,l} (\log J_{l-k})^i_j \, q^j_l \frac{\partial B(\lfloor J[s]\mathbf{q} \rfloor)}{\partial q^i_k} ds \\
&= \int_0^s \frac{\partial}{\partial s} B(\lfloor J[s]\mathbf{q} \rfloor) ds \\
&= B(\lfloor J[s]\mathbf{q} \rfloor) - B(\lfloor J[0]\mathbf{q} \rfloor) \\
&= B(\lfloor J[s]\mathbf{q} \rfloor) - B(\mathbf{q})
\end{aligned}$$

Therefore the second term of $\frac{d}{ds}(\star_r)$ can be written as

$$\frac{1}{2\lambda}\left(B(\mathbf{q}) + \sum_{i,j,k,l} (\log J_{l-k})^i_j \, q^j_l \frac{\partial}{\partial q^i_k} \, W[s](\mathbf{q}) \right) (\star_r).$$

The formulas for the first and second term together yield the following expression for $\frac{d}{ds}(\star_r)$:

$$\sum_{i,j,k,l} (\log J_{l-k})^i_j \, q^j_l \frac{\partial}{\partial q^i_k}(\star_r) + \frac{1}{2\lambda} \sum_{i,j,k,l} \sum_{i'} (-1)^k g_{ii'} \big((\log J)_{k+l+1} \big)^{i'}_j q^i_k q^j_l \, (\star_r).$$

Let $\star_l$ denote the left side of (30). The differential equation,

$$\frac{d}{ds}(\star_l) = -\widehat{\log J}(\star_l)$$

is obtained. The quantization $-\widehat{\log J}$ has been explicitly computed in Lemma 12. After comparison, we conclude the functions $\star_l$ and $\star_r$ satisfy the same differential equation in s. $\square$

The main results of the Section are summarized below. The proofs are obtained from Lemmas 9-13 and Propositions 1 and 2.

Theorem 1. *The τ-function of $\mathcal{F}$ at p' is an element of the Fock space $\mathbb{F}_{\mathbf{t}_{p'}}$ modulo $\mathbb{C}^*$,*

$$\tau^{\mathcal{F}}_{p'}(\lambda, \mathbf{t}_{p'}) \in \mathbb{F}_{\mathbf{t}_{p'}} / \mathbb{C}^*.$$

Moreover, the τ-function of $\mathcal{F}$ is obtained from the product,

$$\gamma(\lambda, \mathbf{q}) = \prod_{i=1}^m \tau_+(\lambda, q^i) \in \mathbb{F}_c \ ,$$

by the following transformation:

$$\tau^{\mathcal{F}}_{p'}(\lambda, \mathbf{t}_{p'}) =$$

$$\left[e^{\frac{1}{2}\int_p^{p'}\sum (R_1)^i_i du^i} e^{\frac{W(\mathbf{q})}{2\lambda}} e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}} \prod_{i=1}^m \tau_+(\lambda, Q^i) \right]_{\mathbf{Q}=R^{-1}\Psi\lfloor J\mathbf{q}\rfloor},$$

via the variable relations

$$\begin{aligned} q^\mu_0 &= \mathbf{t}^\mu_{0,p'} + t^\mu(p'), \\ q^1_1 &= \mathbf{t}^1_{1,p'} - 1, \\ q^\mu_k &= \mathbf{t}^\mu_{k,p'} \quad \textit{otherwise.} \end{aligned} \tag{33}$$

In fact, the scalar ambiguity is only up to a 24^{th} root of unity which can be chosen uniformly for all p' near p.

3.4. M-independence. By definition, the function $\tau^{\mathcal{F}}_{p'}$ depends on the point p' near p (either in an open or a formal neighborhood). The M-dependence of $\tau^{\mathcal{F}}_{p'}$ can been seen in the M-dependence of the variables $\mathbf{t}_{p'}$ and the M-dependence of the series coefficients. Let $\tau^{\mathcal{F}}$ denote the full M-dependent τ-function near p,

The main result of the Section is the calculation of the derivatives,

$$\frac{\partial \tau^{\mathcal{F}}}{\partial u^i},$$

with respect to the canonical coordinates near p. These derivatives are well-defined series, and, in fact, vanish identically near p.

All the derivatives in Section 3.4 are understood to be taken *along the manifold M*. Derivatives of $\tau^{\mathcal{F}}_{p'}$ in the variables $\mathbf{t}_{p'}$ do not play a role. For example, in Lemma 14 below, $\widehat{R}_c$ is viewed as an M-dependent differential operator in the variables $\mathbf{q}$, and

$$d\widehat{R}_c = \sum_i \frac{\partial \widehat{R}_c}{\partial u^i} du^i,$$

is a differential operator in $\mathbf{q}$ with 1-form valued coefficients on M.

Before calculating the derivatives of $\tau^{\mathcal{F}}$, an auxiliary results calculating the derivatives of $\widehat{R}_c$ and $\widehat{J}^{-1}$ will be obtained.

Lemma 14. $(d + \Psi d\Psi^{-1})\widehat{R}_c = [\widehat{(\frac{d\mathbf{u}}{z})}, \widehat{R}_c] - \left(\frac{1}{2}\sum_i (R_1)^i_i d\mathbf{u}^i\right)\widehat{R}_c.$

Proof. By Corollary 1 of Chapter 1, the R-calibration satisfies the following matrix equation in the canonical frame $\{\tilde{\epsilon}_i\}$:

$$(d + \Psi d\Psi^{-1})R = [(\frac{d\mathbf{u}}{z}), R].$$

Therefore,

$$R^{-1}(d+\Psi d\Psi^{-1})R = R^{-1}[(\frac{d\mathbf{u}}{z}),R]. \tag{34}$$

We will quantize both sides of (34) with respect to the canonical frame.

We first study the left side of (34). By equation (15) of Part 1,

$$\Psi d\Psi^{-1} + (\Psi d\Psi^{-1})^t = 0.$$

Hence, the matrix $\Psi d\Psi$ determines an infinitesimal symplectic transformation in the canonical frame. By Lemmas 6 and 7, the quantization of the left side of (34) is well defined,

$$\begin{aligned}\big(R^{-1}dR + R^{-1}\Psi d\Psi^{-1}R\big)\widehat{\ } &= \widehat{R^{-1}dR} + (R^{-1}\Psi d\Psi^{-1}R)\widehat{\ } \\ &= \widehat{R}_c^{-1}d\widehat{R}_c + \widehat{R}_c^{-1}\widehat{\Psi d\Psi^{-1}}\widehat{R}_c \\ &= \widehat{R}_c^{-1}(d+\widehat{\Psi d\Psi^{-1}})\widehat{R}_c.\end{aligned}$$

The second equality follows from Lemmas 7 and 8.

By a straightforward calculation left to the reader,

$$\widehat{\Psi d\Psi^{-1}} = \Psi d\Psi^{-1}, \tag{35}$$

where the operators on the right side of (35) act on the Fock space of functions in the variables $\mathbf{q}$ by the definitions of Section 2.6. We conclude,

$$\big(R^{-1}dR + R^{-1}\Psi d\Psi^{-1}R\big)\widehat{\ } = \widehat{R}_c^{-1}(d+\Psi d\Psi^{-1})\widehat{R}_c. \tag{36}$$

We now consider the right side of (34). The quantization is well defined, and we obtain:

$$\begin{aligned}\big(R^{-1}[\frac{d\mathbf{u}}{z},R]\big)\widehat{\ } &= \big(R^{-1}\frac{d\mathbf{u}}{z}R\big)\widehat{\ } - \widehat{\frac{d\mathbf{u}}{z}} \\ &= \widehat{R}_c^{-1}\ \widehat{\frac{d\mathbf{u}}{z}}\ \widehat{R}_c - \big(\frac{1}{2}\sum_i (R_1)^i_i d\mathbf{u}^i\big) - \widehat{\frac{d\mathbf{u}}{z}} \\ &= \widehat{R}_c^{-1}\Big([\widehat{\frac{d\mathbf{u}}{z}},\widehat{R}_c] - \big(\frac{1}{2}\sum_i (R_1)^i_i d\mathbf{u}^i\big)\widehat{R}_c\Big).\end{aligned}$$

The second equality will be discussed below. The Lemma then follows from equations (34) and (36).

The quantization $(R^{-1}\frac{d\mathbf{u}}{z}R)\widehat{\ }$satisfies all the hypothesis of Lemma 8 except for one. Because of the $1/z$ dependence, the infinitesimal symplectic transformation $\frac{d\mathbf{u}}{z}$ in the center is *not* determined an endomorphism of H. However, since the z dependence is slight, the proof of Lemma 8 may be followed. In fact, the *only* change required in the proof is the replacement of (16) by

$$[\widehat{\log R}, \widehat{\frac{d\mathbf{u}}{z}}] = [\log R, \frac{d\mathbf{u}}{z}]\widehat{\ } + \text{cocycles}. \tag{37}$$

All the further iterated commutators are easily seen to be exactly compatible with quantization.

The only nonvanishing cocycle which appears in (37) is

$$\mathcal{C}\Big(P[R_1 z], P[\frac{d\mathbf{u}}{z}]\Big).$$

After dropping the pq terms,

$$\begin{aligned} P[R_1 z] &= \frac{1}{2}\sum_{ij}(R_1)^i_j p^i_0 p^j_0 + \dots, \\ P[\frac{d\mathbf{u}}{z}] &= -\frac{1}{2}\sum_i du^i (q^i_0)^2 + \dots. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{C}\Big(P[R_1 z], P[\frac{d\mathbf{u}}{z}]\Big) &= \mathcal{C}\Big(\frac{1}{2}\sum_{ij}(R_1)^i_j p^i_0 p^j_0, -\frac{1}{2}\sum_i du^i (q^i_0)^2\Big) \\ &= -\frac{1}{2}\sum_i (R_1)^i_i du^i. \end{aligned}$$

After replacing (16) by (37) in the proof of Lemma 8, we find

$$(R^{-1}\frac{d\mathbf{u}}{z}R)\widehat{\ } = \widehat{R}_c^{-1}\ \widehat{\frac{d\mathbf{u}}{z}}\ \widehat{R}_c - \Big(\frac{1}{2}\sum_i (R_1)^i_i d\mathbf{u}^i\Big),$$

completing the proof. □

Let A^c_i denote the quantum multiplication operator $\epsilon_i *$ where

$$\epsilon_i = \frac{\partial}{\partial u^i}.$$

The derivative of $\widehat{J}^{-1}$ is determined by the following Lemma.

Lemma 15. $\frac{\partial \widehat{J}^{-1}}{\partial u^i} = -\widehat{J}^{-1}\widehat{\frac{A^c_i}{z}}$.

Proof. The flatness equation (10) may be written as

$$\frac{\partial J}{\partial u^i} = \frac{A^c_i}{z} J,$$

or equivalently,

$$J\ \frac{\partial J^{-1}}{\partial u^i} = -\frac{A^c_i}{z}.$$

After quantization, we obtain

$$\widehat{J}\,\frac{\partial \widehat{J}^{-1}}{\partial u^i} = -\frac{\widehat{A^c_i}}{z},$$

by Lemma 7. $\square$

PROPOSITION 3. *The derivatives of the τ-function vanish near p,*

$$\frac{\partial \tau^{\mathcal{F}}}{\partial u^i} = 0.$$

PROOF. We write $\tau^{\mathcal{F}}$ in the following form near p:

$$\tau^{\mathcal{F}}_u(\lambda, \mathbf{t}) = \left[e^{\frac{1}{2}\int_p^u \sum (R_1)^j_j du^j}\,\widehat{J}^{-1}\Psi^{-1}\widehat{R}_c \prod_{j=1}^m \tau_+(\lambda, q^j)\right]_{\mathbf{q}_+=\mathbf{t}}.$$

We will now compute the derivative $\partial_{u_i}\tau^{\mathcal{F}}$ via the Leibniz rule. The calculation is separated into three parts. First we differentiate the exponential prefactor,

$$\partial_{u_i} e^{\frac{1}{2}\int_p^u \sum_j (R_1)^j_j du^j} = \frac{1}{2}(R_1)^i_i\; e^{\frac{1}{2}\int_p^u \sum_j (R_1)^j_j du^j}.$$

Second, we differential the product $\widehat{J}^{-1}\Psi^{-1}$,

$$\begin{aligned}
\partial_{u_i}(\widehat{J}^{-1}\Psi^{-1}) &= (\partial_{u_i}\widehat{J}^{-1})\Psi^{-1} + \widehat{J}^{-1}(\partial_{u_i}\Psi^{-1}) \\
&= -\widehat{J}^{-1}\frac{\widehat{A^c_i}}{z}\Psi^{-1} + \widehat{J}^{-1}\partial_{u_i}\Psi^{-1} \\
&= \widehat{J}^{-1}\Psi^{-1}\left(\Psi\partial_{u_i}\Psi^{-1} - \frac{\widehat{d\mathbf{u}(\partial_{u_i})}}{z}\right).
\end{aligned}$$

Lemma 15 is used in the second equality. The last equality above depends upon the idempotence of quantum multiplication by ∂_{u_i} in the canonical frame. Finally,

$$\partial_{u_i}\widehat{R}_c = -(\Psi\partial_{u_i}\Psi^{-1})\widehat{R}_c + [\frac{\widehat{d\mathbf{u}(\partial_{u_i})}}{z}, \widehat{R}_c] - \frac{1}{2}(R_1)^i_i\;\widehat{R}_c, \tag{38}$$

by Lemma 14.

The Leibniz rule expresses $\partial_{u_i}\tau^{\mathcal{F}}$ as a sum of three terms corresponding to the above derivative calculations. We may replace the right side of (38) by

$$\Big(- \Psi\partial_{u_i}\Psi^{-1} + \frac{\widehat{d\mathbf{u}(\partial_{u_i})}}{z} - \frac{1}{2}(R_1)^i_i\Big)\widehat{R}_c$$

in the Leibniz expansion since the omitted term annihilates

$$\prod_{j=1}^{m} \tau_+(\lambda, q^j)$$

by the string equation. Then, the three terms of the Leibniz expansion sum to zero. We conclude $\partial_{u_i}\tau^{\mathcal{F}}$ vanishes. $\square$

In case $\mathcal{F}$ is a complex Frobenius manifold, a stronger form of M-invariance will be discussed in Section 4.2.

CHAPTER 10

Descendent potentials

1. Definitions

Let $\mathcal{F}$ be a complex or formal Frobenius manifold of dimension m with R and J-calibrations. Let p be a semisimple point of $\mathcal{F}$. Define the genus g descendent potential $\mathbf{F}_{g,p'}$ of $\mathcal{F}$ at p' near p by:

$$\sum_{g\in\mathbb{Z}} \lambda^{g-1}\mathbf{F}^{\mathcal{F}}_{g,p'}(\mathbf{t}_{p'}) = \log \tau^{\mathcal{F}}_{p'}(\lambda, \mathbf{t}_{p'}).$$

Since $\tau^{\mathcal{F}}_{p}(\lambda, \mathbf{t}_{p'})$ contains a scalar ambiguity, $\mathbf{F}^{\mathcal{F}}_{1,p'}$ is only determined up to an additive constant.

LEMMA 16. *$\mathbf{F}^{\mathcal{F}}_{g,p'}(\mathbf{t}_{p'}) = 0$ for $g < 0$.*

PROOF. We write $\tau^{\mathcal{F}}_{p'}(\lambda, \mathbf{t}_{p'})$ in the form of Theorem 1:

$$\left[e^{\frac{1}{2}\int_p^{p'} \sum (R_1)^i_i du^i} e^{\frac{W(\mathbf{q})}{2\lambda}} e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}} \prod_{i=1}^m \tau_+(\lambda, Q^i) \right]_{\mathbf{Q}=R^{-1}\Psi\lfloor J\mathbf{q}\rfloor},$$

via the variable relations (33).

The change of variables $\mathbf{Q} = R^{-1}\Psi\lfloor J\mathbf{q}\rfloor$ does *not* involve λ. The multiplicative prefactors,

$$e^{\frac{1}{2}\int_p^{p'} \sum (R_1)^i_i du^i} e^{\frac{W(\mathbf{q})}{2\lambda}},$$

contribute summands to $\log \tau^{\mathcal{F}}_{p'}$ in genus 0 and 1. The main term,

$$e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}} \prod_{i=1}^m \tau_+(\lambda, Q^i), \tag{39}$$

is analyzed via Wick expansion.

Let Γ be a graph arising in the Wick expansion of (39). Let V and E be the vertex and edge sets of Γ. The vertices $v \in V$ carry a genus g_v and an index assignment $j(v)$. To each vertex v, we associate the genus g_v potential

$$\lambda^{g(v)-1}F_{g(v)+}(Q^{j(v)})$$

defined in Section 3.2 of Chapter 8. Each edge connecting v_1 and v_2 corresponds to an application of the the quadratic differential operator,

$$\frac{\lambda}{2} E_{kl}^{j(v_1)j(v_2)} \frac{\partial}{\partial Q_k^{j(v_1)}} \frac{\partial}{\partial Q_l^{j(v_2)}},$$

to the potentials on the vertices

The graph Γ contributes to the coefficient of $\lambda^{g(\Gamma)-1}$ of (39), where the graph genus $g(\Gamma)$ is defined by:

$$g_\Gamma = \sum_{v \in V} g_v + |E| - |V| + 1.$$

The Wick expansion of

$$\log \left(e^{\frac{\lambda}{2} \sum E_{kl}^{ij} \frac{\partial}{\partial Q_k^i} \frac{\partial}{\partial Q_l^j}} \prod_{i=1}^{m} \tau_+(\lambda, Q^i) \right) \tag{40}$$

is equal to the contribution of the *connected* graphs Γ. Since

$$g_\Gamma \geq 0$$

for a connected graph, the logarithm (40) contains only terms with $\lambda^{g-1 \geq -1}$. □

2. The Gromov-Witten J-calibration

2.1. $\mathcal{F}(X)$. Let X be a nonsingular projective variety with even cohomology. Let $\mathcal{F}(X)$ be the formal Frobenius manifold associated to X in Chapter 2.

We briefly review the notation of Chapter 2. Let $T_1, \ldots, T_m$ be a basis of $H^*(X, \mathbb{C})$ with T_1 equal to the identity $\mathbf{1}$. Let

$$t^1, \ldots, t^m, \quad \partial_1, \ldots, \partial_m,$$

be the associated flat coordinates and flat vector fields. The metric g is determined by:

$$g_{ab} = \int_X T_a \cup T_b.$$

The Frobenius structure is obtained from the genus 0 Gromov-Witten potential F_0^X for primary fields.

The descendent Gromov-Witten theory of X provides a canonical J-calibration of $\mathcal{F}(X)$ at the origin. The Gromov-Witten J-calibration motivates the more general definitions of Section 3.

The Gromov-Witten J-calibration can be written in the following form:

$$J_i^j = \delta_i^j + \sum_k g^{jk} \langle\langle T_k, \frac{T_i}{z - \psi} \rangle\rangle_0^X, \tag{41}$$

We use here the double bracket notation of Chapter 4:

$$\langle\langle T_k, \frac{T_i}{z-\psi}\rangle\rangle_0^X = \sum_{n\geq 0,\ \beta,\ (n,\beta)\neq(0,0)} \frac{Q^\beta}{n!}\langle T_k, \frac{T_i}{z-\psi}, \gamma, \ldots, \gamma\rangle_{0,2+n,\beta}^X,$$

where $1 \leq i, k \leq m$ and $\gamma = \sum_\mu t^\mu T_\mu$. Here, Q is the Novikov parameter.

2.2. The functions W_{kl}^{ij}. The functions W_{kl}^{ij} on $\mathcal{F}(X)$ are obtained from the Gromov-Witten J-calibration by definition (28).

Lemma 17. $\sum_{k,l=0}^\infty W_{kl}^{ij} w^{-k} z^{-l} = wz\langle\langle \frac{T_i}{w-\psi}, \frac{T_j}{z-\psi}\rangle\rangle_0^X$.

Proof. After expanding definition (28) using the formula for the Gromov-Witten J-calibration (41), we obtain,

$$\begin{aligned}\left(\frac{1}{w}+\frac{1}{z}\right)\sum_{k,l=0}^\infty W_{kl}^{ij} w^{-k} z^{-l} &= \sum_{a,b}\langle\langle \frac{T_i}{w-\psi}, T_a\rangle\rangle_0^X g^{ab}\langle\langle T_b, \frac{T_j}{z-\psi}\rangle\rangle_0^X \\ &\quad \langle\langle \frac{T_i}{w-\psi}, T_j\rangle\rangle_0^X + \langle\langle T_i, \frac{T_j}{z-\psi}\rangle\rangle_0^X.\end{aligned}$$

The Lemma is then a straightforward consequence of the WDVV equation:

$$\begin{aligned}&\sum_{a,b}\langle\langle \frac{T_i}{w-\psi}, \mathbf{1}, T_a\rangle\rangle_0^X g^{ab}\langle\langle T_b, \frac{T_j}{z-\psi}, \mathbf{1}\rangle\rangle_0^X = \\ &\qquad\qquad \sum_{a,b}\langle\langle \frac{T_i}{w-\psi}, \frac{T_j}{z-\psi}, T_a\rangle\rangle_0^X g^{ab}\langle\langle T_b, \mathbf{1}, \mathbf{1}\rangle\rangle_0^X.\end{aligned}$$

Expansion of the left side using the string equation yields

$$\frac{1}{wz}\left(\frac{1}{w}+\frac{1}{z}\right)\sum_{k,l=0}^\infty W_{kl}^{ij} w^{-k} z^{-l} + \frac{g_{ij}}{wz}.$$

By the axiom of the fundamental class and the string equation, the right side equals

$$\left(\frac{1}{w}+\frac{1}{z}\right)\langle\langle \frac{T_i}{w-\psi}, \frac{T_j}{z-\psi}\rangle\rangle_0^X + \frac{g_{ij}}{wz},$$

completing the proof. □

2.3. Descendents. Let $\mathbf{t} = \{\mathbf{t}^i_k\}$ be the set of descendent variables. The genus 0 descendent potential of X is defined by:

$$\mathbf{F}_0^X(Q, \mathbf{t}) = \sum_{n \geq 0} \sum_{\beta} \frac{Q^\beta}{n!} \langle \underbrace{\mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)}_{n} \rangle^X_{0,n,\beta},$$

where

$$\mathbf{t}(\psi) = \sum_{i,k} \mathbf{t}^i_k T_i \psi^k.$$

We will express the descendent potential $\mathbf{F}_0^X$ in terms of *non-descendent* double brackets.

The first step is to consider the critical points of the function

$$f(Q, t, \mathbf{t}) = \langle\langle \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle\rangle^X_0.$$

We view f as functions in the variables t with parameters $\mathbf{t}$. The dependence of f on the descendent variables $\mathbf{t}$ is linear. The *critical points* of f occur when

$$\frac{\partial f}{\partial t^i} = 0,$$

for all i.

LEMMA 18. *There is a unique critical point $t^i_{crit} \in \mathbb{C}[[\mathbf{t}]]$ satisfying*

$$t^i_{crit}|_{\mathbf{t}^i_{k>0}=0} = \mathbf{t}^i_0.$$

PROOF. The critical point equations are, for all i',

$$\langle\langle \mathbf{1}, T_{i'}, \mathbf{t}(\psi) - \psi \rangle\rangle^X_0 = 0.$$

After raising the first index, we find,

$$\sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \psi \rangle\rangle^X_0 = \sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \mathbf{t}(\psi) \rangle\rangle^X_0.$$

By applications of the string equation and the axiom of the fundamental class, the left side is equal to t^i. Hence, the critical point equations are:

$$t^i = \sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \mathbf{t}(\psi) \rangle\rangle^X_0. \tag{42}$$

The right side is *homogeneous* of degree 1 in the variables $\mathbf{t}$.

After the restriction $\mathbf{t}^i_{k>0} = 0$, the right side of (42) equals $\mathbf{t}^i_0$ by the axiom of the fundamental class. The solution $t^i = \mathbf{t}^i_0$ of the restricted system admits a unique prolongation to a solution of the full system (42). The prolongation is obtained by iteration of the equations (42). □

The following result, due to Dubrovin [Du], motivates the analysis of genus 0 potentials of Frobenius manifolds in Section 3.

Lemma 19. $\mathbf{F}_0^X = \frac{1}{2}\langle\langle \mathbf{t}(\psi) - \psi, \mathbf{t}(\psi) - \psi \rangle\rangle_0^X|_{t=t_{crit}(\mathbf{t})}$.

Proof. An elementary argument using the dilaton equation yields the following relation:

$$\mathbf{F}_0^X = \frac{1}{2}\sum_{i,k,i',k'} (\mathbf{t}_k^i - \delta^{i0}\delta_{k1})(\mathbf{t}_{k'}^{i'} - \delta^{i'0}\delta_{k'1})\frac{\partial^2}{\partial \mathbf{t}_k^i \partial \mathbf{t}_{k'}^{i'}}\mathbf{F}_0^X.$$

Hence, the equality,

$$\frac{\partial^2}{\partial \mathbf{t}_k^i \partial \mathbf{t}_{k'}^{i'}}\mathbf{F}_0^X = \langle\langle T_i\psi^k, T_{i'}\psi^{k'}\rangle\rangle_0^X|_{t=t_{crit}(\mathbf{t})} \tag{43}$$

implies the Lemma.

Equation (43) is true by definition after the restriction $\mathbf{t}_{k>0}^i = 0$. By the topological recursion relation, for $l > 0$,

$$\frac{\partial^3 \mathbf{F}_0^X}{\partial \mathbf{t}_l^j \partial \mathbf{t}_k^i \partial \mathbf{t}_{k'}^{i'}} = \sum_{a,b} \frac{\partial^3 \mathbf{F}_0^X}{\partial \mathbf{t}_k^i \partial \mathbf{t}_{k'}^{i'} \partial \mathbf{t}_0^a} g^{ab} \frac{\partial^2 \mathbf{F}_0^X}{\partial \mathbf{t}_0^b \partial \mathbf{t}_{l-1}^j}.$$

Using the chain rule and equation (42), we derive an analogous equation for the right side of (43),

$$\frac{\partial}{\partial \mathbf{t}_l^j}\ \langle\langle T_i\psi^k, T_{i'}\psi^{k'}\rangle\rangle_0^X|_{t=t_{crit}(\mathbf{t})} =$$

$$\sum_{a,b}\langle\langle T_i\psi^k, T_{i'}\psi^{k'}, T_a\rangle\rangle_0^X g^{ab} \langle\langle T_b, T_j\psi^{l-1}\rangle\rangle_0^X\ |_{t=t_{crit}(\mathbf{t})}.$$

Equation (43) is then proven order by order in the descendent variables $\mathbf{t}_{k>0}^i$. □

3. Genus 0 and 1

3.1. Double brackets. Let $\mathcal{F}$ be a complex or formal Frobenius manifold of dimension m with R and J-calibrations. Let p be a semisimple point of $\mathcal{F}$ with tangent space TM_p. Let

$$T_1, \ldots, T_m$$

denote the values of the vectors fields $\{\partial_i\}$ at p.

Motivated by Lemma 17, we *define* the double bracket of $\mathcal{F}$ at p by

$$\langle\langle T_i\psi^k, T_j\psi^l\rangle\rangle_0^{\mathcal{F}} = W_{kl}^{ij}(t).$$

The functions $W_{kl}^{ij}(t)$ are holomorphic near $t = t(p)$ if $\mathcal{F}$ is complex. If $\mathcal{F}$ is formal, the functions $W_{kl}^{ij}(t)$ are power series.

By setting the variable $w^{-1}=0$ in the definition of the functions W^{ij}_{kl}, we find,

$$J_i^j = \delta_i^j + \sum_k g^{jk}\langle\langle T_k, \frac{T_i}{z-\psi}\rangle\rangle_0^{\mathcal{F}}, \tag{44}$$

in agreement with (41).

Additional insertions in the double bracket are defined by differentiation:

$$\langle\langle T_i\psi^k, T_{a_1},\ldots,T_{a_n}, T_j\psi^l\rangle\rangle_0^{\mathcal{F}} = \frac{\partial^n}{\partial t^{a_1}\ldots\partial t^{a_n}} W^{ij}_{kl}(t).$$

The symmetry $W^{ij}_{kl} = W^{ji}_{lk}$ immediately implies

$$\langle\langle T_i\psi^k, T_j\psi^l\rangle\rangle_0^{\mathcal{F}} = \langle\langle T_j\psi^l, T_i\psi^k\rangle\rangle_0^{\mathcal{F}}.$$

The following symmetry, however, is more subtle.

Lemma 20. $\langle\langle T_i, T_a, T_j\psi^l\rangle\rangle_0^{\mathcal{F}} = \langle\langle T_a, T_i, T_j\psi^l\rangle\rangle_0^{\mathcal{F}}$.

Proof. Let $V_r(z)$ denote the vector field obtained from the r^{th} column of the J-calibration,

$$V_r(z) = \sum_s J_r^s(z)\partial_s.$$

The column vectors are flat with respect to the Dubrovin connection,

$$\frac{\partial}{\partial t^a}V_r(z) = \frac{1}{z}\partial_a * V_r(z), \tag{45}$$

and satisfy $V_r(z) = \partial_r$ when $z^{-1}=0$.

Equation (28) can be reformulated in a more convenient form:

$$\frac{\langle V_i(w), V_j(z)\rangle - g_{ij}}{w^{-1}+z^{-1}} = \sum_{k,l=0}^{\infty} W^{ij}_{kl} w^{-k} z^{-l}.$$

Then, by definition,

$$\frac{1}{w^{-1}+z^{-1}}\frac{\partial}{\partial t^a}\langle V_i(w), V_j(z)\rangle = \sum_{k,l=0}^{\infty}\langle\langle T_i\psi^k, T_a, T_j\psi^l\rangle\rangle_0^{\mathcal{F}} w^{-k}z^{-l}.$$

Using the flatness of the coordinates, property (45), and associativity, we find:

$$\begin{aligned}
\frac{\partial}{\partial t^a}\langle V_i(w), V_j(z)\rangle &= \langle\frac{\partial}{\partial t^a}V_i(w), V_j(z)\rangle + \langle V_i(w), \frac{\partial}{\partial t^a}V_j(z)\rangle \\
&= \frac{1}{w}\langle\partial_a * V_i(w), V_j(z)\rangle + \frac{1}{z}\langle V_i(w), \partial_a * V_j(z)\rangle \\
&= \left(\frac{1}{w}+\frac{1}{z}\right)\ \langle V_i(w), \partial_a * V_j(z)\rangle.
\end{aligned}$$

After the restriction $w^{-1} = 0$, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} \langle\langle T_i, T_a, T_j\psi^l \rangle\rangle_0^{\mathcal{F}} z^{-l+1} &= \frac{1}{z}\langle \partial_i, \partial_a * V_j(z)\rangle \\ &= \frac{1}{z}\langle \partial_a, \partial_i * V_j(z)\rangle \\ &= \sum_{l=0}^{\infty} \langle\langle T_i, T_a, T_j\psi^l \rangle\rangle_0^{\mathcal{F}} z^{-l+1},\end{aligned}$$

proving the Lemma. □

The methods of the proof of Lemma 20 also yield the basic string equation,

$$\sum_{k,l=0}^{\infty} \langle\langle T_i\psi^k, \mathbf{1}, T_j\psi^l \rangle\rangle_0^{\mathcal{F}} w^{-k} z^{-l} = g_{ij} + \left(\frac{1}{w} + \frac{1}{z}\right) \sum_{k,l=0}^{\infty} \langle\langle T_i\psi^k, T_j\psi^l \rangle\rangle_0^{\mathcal{F}}.$$

3.2. Critical points. We consider the critical points of the function,

$$f(t, \mathbf{t}) = \langle\langle \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle\rangle_0^{\mathcal{F}},$$

in the variables t with parameters $\mathbf{t}$.

Lemma 21. *There is a unique critical point $t^i_{crit,p} \in \mathbb{C}[[\mathbf{t}_p]]$ satisfying*

$$t^i_{crit,p}|_{\mathbf{t}^i_{k>0}=0} = t^i(p) + (\mathbf{t}^i_0 - t^i(p)).$$

Proof. We follow the proof of Lemma 18. The (raised) critical point equations are:

$$\sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \psi \rangle\rangle_0^{\mathcal{F}} = \sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \mathbf{t}(\psi) \rangle\rangle_0^{\mathcal{F}}.$$

We first analyze the left side:

$$\begin{aligned}\sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \psi \rangle\rangle_0^{\mathcal{F}} &= \sum_{i'} g^{ii'} \langle\langle T_{i'}, \mathbf{1}, \psi \rangle\rangle_0^{\mathcal{F}} \\ &= \sum_{i'} g^{ii'} \langle\langle T_{i'}, \mathbf{1} \rangle\rangle_0^{\mathcal{F}} \\ &= t^i\end{aligned}$$

The first two equalities follow from Lemma 20 and the string equation. The last equality is obtained from the definition of the double bracket and the normalization property of the J-calibration.

The critical point equations can be written as:

$$t^i = \sum_{i'} g^{ii'} \langle\langle \mathbf{1}, T_{i'}, \mathbf{t}(\psi) \rangle\rangle_0^{X}. \tag{46}$$

The right side is *homogeneous* of degree 1 in the variables $\mathbf{t}$.

After the restriction $\mathbf{t}^i_{k>0} = 0$, the right side of (46) equals $\mathbf{t}^i_0$ by the normalization condition. The solution $t^i = \mathbf{t}^i_0$ of the restricted system admits a unique prolongation by iteration to a solution of the full system.

The iteration procedure yields a series, $t_{crit,p}$, in the variables $\mathbf{t}_p$,

$$\begin{aligned} \mathbf{t}^\mu_{0,p} &= \mathbf{t}^\mu_0 - t^\mu(p), \\ \mathbf{t}^\mu_{k\neq 0,p} &= \mathbf{t}^\mu_k, \end{aligned}$$

since the functions W^{ij}_{kl} are defined near p. $\square$

3.3. Genus 0. We now analyze the genus 0 descendent potential $\mathbf{F}^{\mathcal{F}}_0$ defined in Section 1. The main result is motivated by Lemma 18.

PROPOSITION 4. $\mathbf{F}^{\mathcal{F}}_{0,p} = \frac{1}{2}\langle\langle \mathbf{t}(\psi) - \psi, \mathbf{t}(\psi) - \psi\rangle\rangle^{\mathcal{F}}_0 |_{t=t_{crit,p}(\mathbf{t}_p)}$,

PROOF. Let p_{crit} be the point of $\mathcal{F}$ with coordinates

$$t^i(p_{crit}) = t^i_{crit,p}(\mathbf{t}_p).$$

The point p_{crit} is *formal* in the variables $\mathbf{t}_p$ and infinitely near p. By the M-invariance of $\tau^{\mathcal{F}}$ we find,

$$\tau^{\mathcal{F}}_p = \tau^{\mathcal{F}}_{p_{crit}}.$$

The Proposition is proved by studying $\tau^{\mathcal{F}}$ at p_{crit}:

$$\left[e^{\frac{1}{2}\int_p^{p_{crit}} \sum (R_1)^i_i du^i} e^{\frac{W(\mathbf{q})}{2\lambda}} e^{\frac{\lambda}{2}\sum E^{ij}_{kl} \frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}} \prod_{i=1}^m \tau_+(\lambda, Q^i) \right]_{\mathbf{Q}=R^{-1}\Psi\lfloor J\mathbf{q}\rfloor},$$

via the change of variables (33). All the function evaluations on M are taken at p_{crit}.

We examine the change of variables relating $\mathbf{Q}$ to $\mathbf{q}$. The first step is $\lfloor J\mathbf{q}\rfloor$,

$$\begin{aligned} \lfloor J\mathbf{q}\rfloor^i_0 &= \sum_{l=0}^{\infty}\sum_j (J_l)^i_j q^j_l \\ &= q^i_0 + \sum_{l>0}\sum_j\sum_{i'} g^{ii'}\langle\langle T_{i'}, T_j\psi^{l-1}\rangle\rangle^{\mathcal{F}}_0 q^j_l \\ &= \sum_{i'} g^{ii'}\langle\langle T_{i'}, \mathbf{1}, \mathbf{t}(\psi)-\psi\rangle\rangle^{\mathcal{F}}_0 \\ &= \sum_{i'} g^{ii'}\langle\langle \mathbf{1}, T_{i'}, \mathbf{t}(\psi)-\psi\rangle\rangle^{\mathcal{F}}_0 \\ &= 0. \end{aligned} \tag{47}$$

The first equality is by definition. The second is obtained from (44). The third uses the string equation and the variable relationship

$$\begin{aligned} q_1^1 + 1 &= \mathbf{t}_1^1, \\ q_{k\neq 1}^{\mu\neq 1} &= \mathbf{t}_k^\mu, . \end{aligned}$$

The fourth is a consequence of Lemma 20. The final vanishing follows from the critical point equation.

Since the transformation $R^{-1}\Psi$ preserves the descendent index 0 variables, we conclude

$$Q_0^i = 0$$

under the variable change $\mathbf{Q} = R^{-1}\Psi\lfloor J\mathbf{q}\rfloor$.

Next, we consider the Wick expansion by connected graphs of

$$\log\left(e^{\frac{\lambda}{2}\sum E_{kl}^{ij}\frac{\partial}{\partial Q_k^i}\frac{\partial}{\partial Q_l^j}} \prod_{i=1}^m \tau_+(\lambda, Q^i)\right) \tag{48}$$

after the evaluation $Q_0^i = 0$. If a connected graph Γ satisfies

$$g_\gamma = 0,$$

Γ must have a vertex v of genus 0 and valence at most two. The Wick contribution of Γ then contains the factor

$$\frac{\partial}{\partial Q_k^{j(v)}}\frac{\partial}{\partial Q_l^{j(v)}} F_{0+}(Q^{j(v)}), \tag{49}$$

where the two derivatives correspond to the two edges incident to v. The factor (49) *vanishes* after the evaluation

$$Q_0^{j(v)} = 0$$

since $F_{0+}(Q^{j(v)})$ is divisible by $(Q_0^{j(v)})^3$. We have proven the logarithm (48) makes no contribution $\mathbf{F}_0^{\mathcal{F}}$.

We conclude the genus 0 term of $\log \tau_{p_{crit}}^{\mathcal{F}}$ is obtained entirely from the prefactors,

$$\begin{aligned} \mathbf{F}_{0,p}^{\mathcal{F}} &= \frac{1}{2}W(\mathbf{q}) \\ &= \frac{1}{2}\langle\langle \mathbf{t}(\psi) - \psi, \mathbf{t}(\psi) - \psi\rangle\rangle_0^{\mathcal{F}}|_{t=t_{crit,p}(\mathbf{t}_p)}. \end{aligned}$$

The proof of the Proposition is complete. □

3.4. Genus 1. An identification of the genus 1 descendent potential is obtained by a similar argument.

PROPOSITION 5. *Modulo an additive constant,*

$$\mathbf{F}^{\mathcal{F}}_{1,p}(\mathbf{t}_p) = \sum_i \frac{1}{48} \log \mathbf{D}^i + \frac{1}{2} \int_p^{p_{crit}} (R_1)^i_i du^i,$$

where

$$\frac{1}{\sqrt{\mathbf{D}^i}} = -\langle\langle \tilde{\epsilon}_i, \mathbf{1}, \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle\rangle^{\mathcal{F}}_0 |_{t=t_{crit,p}(\mathbf{t}_p)}.$$

PROOF. We follow the proof of Proposition 4. After the evaluation

$$Q^i_0 = 0,$$

there are exactly m connected graphs Γ satisfying $g_\Gamma = 1$ which contribute to (48). The i^{th} graph has a single vertex of v of genus $g_v = 1$ with index assignment $j(v) = i$ an *no* edges. Dimension constraints force the contribution of the i^{th} graph to (48) to be:

$$\begin{aligned} F_{1+}(Q^i)|_{Q^i_0=0} &= \sum_{n=0}^{\infty} \frac{(q_1+1)^n}{n!} \int_{\overline{M}_{1,n}} \psi_1 \cdots \psi_n \\ &= -\frac{1}{24} \log -Q^i_1. \end{aligned}$$

Since $\lfloor J\mathbf{q} \rfloor^i_0 = 0$ at the critical point and R_0 is the identity,

$$Q^i_1 = \sum_j \Psi^i_j \lfloor J\mathbf{q} \rfloor^j_1.$$

By a calculation parallel to (47),

$$\lfloor J\mathbf{q} \rfloor^j_1 = \sum_{j'} g^{jj'} \langle\langle T_{j'}, \mathbf{1}, \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle\rangle^{\mathcal{F}}_0 .$$

After the change of basis Ψ, we find

$$Q^i_1 = \langle\langle \tilde{\epsilon}_i, \mathbf{1}, \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle\rangle^{\mathcal{F}}_0 .$$

We conclude the contribution of (48) to $\mathbf{F}^{\mathcal{F}}_{1,p}$ is:

$$-\frac{1}{24} \log -\langle\langle \tilde{\epsilon}_i, \mathbf{1}, \mathbf{1}, \mathbf{t}(\psi) - \psi \rangle\rangle^{\mathcal{F}}_0 .$$

The proof of the Proposition is completed by including the contribution to $\log \tau^{\mathcal{F}}_{p_{crit}}$ of the prefactor $\exp(\frac{1}{2} \int_p^{p_{crit}} (R_1)^i_i du^i)$. $\square$

We may now match the genus 1 formula stated in Section 3 of Chapter 6. Because of the additive ambiguity in genus 1, the differential of the potential is a more natural object.

The restriction $\mathbf{t} \to t$ from descendent variables to flat coordinates on the Frobenius manifold is defined by:

$$\begin{aligned} \mathbf{t}_0^i &= t_0^i, \\ \mathbf{t}_{k>0}^i &= 0. \end{aligned}$$

Let $F_{1,p}$ denote the restriction to flat coordinates of $\mathbf{F}_{1,p}^{\mathcal{F}}$.

COROLLARY 1. *After restriction to flat coordinates,*

$$dF_1 = \sum_i \frac{1}{48} \log \Delta_i + \frac{1}{2}(R_1)_i^i du^i.$$

PROOF. The proof is obtained from Proposition 5 by a straightforward calculation. For the first term,

$$\begin{aligned} \frac{1}{\sqrt{\mathbf{D}^i}} \,|_{\mathbf{t}\to t} &= \langle\langle \tilde{\epsilon}_i, \mathbf{1}, \mathbf{1}, \psi - \mathbf{t}(\psi) \rangle\rangle_0^{\mathcal{F}} |_{\mathbf{t}\to t} \\ &= \langle\langle \tilde{\epsilon}_i, \mathbf{1}, \mathbf{1}, \psi \rangle\rangle_0^{\mathcal{F}} \\ &= \langle\langle \tilde{\epsilon}_i, \mathbf{1}, \mathbf{1}, \rangle\rangle_0^{\mathcal{F}} \\ &= \langle \tilde{\epsilon}_i, 1 \rangle \\ &= \langle \tilde{\epsilon}_i, \sum_j \epsilon_j \rangle \\ &= \frac{1}{\sqrt{\Delta^i}}. \end{aligned}$$

The second term, after restriction, equals

$$\frac{1}{2} \int_p^{p_t} (R_1)_i^i du^i,$$

where $t^i(p_t) = t^i$. The differential is then easily obtained. □

4. Higher genus

4.1. Potentials. The formulas for the higher genus descendent potentials generalize the formulas of Chapter 6 for primary fields.

THEOREM 2. *The higher genus descendent potential are determined by:*

$$\exp\left(\sum_{g\geq 2} \lambda^{g-1} \mathbf{F}_{g,p}^{\mathcal{F}}(\mathbf{t}_p)\right) =$$
$$\left[\exp\Big(\frac{\lambda}{2} \sum_{k,l=0}^{\infty} \sum_{i,j} E_{kl}^{ij} \sqrt{\mathbf{D}^i}\sqrt{\mathbf{D}^j} \frac{\partial}{\partial Q_k^i} \frac{\partial}{\partial Q_l^j}\Big) \prod_m \tau(\lambda \mathbf{D}^m, Q^m)\right]_{Q_k^i = \mathbf{T}_k^i},$$

where $\mathbf{T}^i_k$ *is defined by the following equality:*

$$\frac{1}{\sqrt{\mathbf{D}^i}}(1-\sum_{k=2}^{\infty}\mathbf{T}^i_k(-z)^{k-1}) =$$
$$\left[-\left(\sum_{k,j}(-z)^{k-1}\langle\langle\tilde{\epsilon}_j, \mathbf{1}^{k+1}, \mathbf{t}(\psi)-\psi\rangle\rangle\right)R^j_i(z)\right]_{t=t_{crit,p}(\mathbf{t}_p)}.$$

Proof. Since the prefactors of $\tau^{\mathcal{F}}_{p_{crit}}$ contribute only to genus 0 and 1, the higher genus descendent potentials are obtained from

$$\log\left[e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}}\prod_{i=1}^{m}\tau_+(\lambda, Q^i)\right]_{\mathbf{Q}=R^{-1}\Psi\lfloor J\mathbf{q}\rfloor}, \tag{50}$$

The logarithm (50), however, has genus 1 contributions by the analysis of Proposition 5. The strategy of the proof is to *remove* the genus 1 contribution of (50) by a change of variables.

We introduce a new set of variables $\tilde{\mathbf{Q}}$ related to the variables $\mathbf{Q}$ by scaling,

$$\tilde{Q}^i_k = \sqrt{\mathbf{D}^i}Q^i_k.$$

Let $\tilde{\mathbf{Q}} = \sqrt{\mathbf{D}}\mathbf{Q}$ denote the scaling transformation.

Let $F_{g+}(q)$ be the potential for connected, genus g curves defined in Section 3.2. If $g \neq 1$, by the dilation equation,

$$\left(\sum_k q_k\frac{\partial}{\partial q_k}\right)\mathcal{F}_g = (2-2g)\mathcal{F}_g,$$

$F_{g+}(q)$ is a *homogeneous* function of q of degree $2-2g$. If $g=1$, the dilaton equation has a inhomogeneous term obtained from the *nonexistence* of $\overline{M}_{1,0}$. However,

$$\frac{1}{24}\log(-q_1) + F_{1+}(q)$$

is homogeneous of degree 0. We conclude,

$$\tau_+(\lambda, Q^m) = (\mathbf{D}^i)^{\frac{1}{48}}\tau_+(\lambda\mathbf{D}^m, \tilde{Q}^m).$$

We rewrite the logarithm (50) in the scaled variables $\tilde{\mathbf{Q}}$:

$$\log\left[e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\sqrt{\mathbf{D}^i}\sqrt{\mathbf{D}^j}\frac{\partial}{\partial \tilde{Q}^i_k}\frac{\partial}{\partial \tilde{Q}^j_l}}\prod_{m}(\mathbf{D}^m)^{\frac{1}{48}}\tau_+(\lambda\mathbf{D}^m, \tilde{Q}^m)\right]_{\tilde{\mathbf{Q}}=\sqrt{\mathbf{D}}R^{-1}\Psi\lfloor J\mathbf{q}\rfloor}.$$

The quadratic differential operator transforms via the chain rule.

Since $\tilde{Q}^i_0 = 0$ and $\tilde{Q}^i_1 = -1$, the logarithm (4.1) has *no* contributions in genus 0. The contribution in genus 1 is obtained only from the factors

$$\prod_m (\mathbf{D}^m)^{\frac{1}{48}}.$$

Moreover these factor do not contribute to the higher genus potentials in the logarithm.

We have proven the *higher genus* contributions of the logarithm (50) and

$$\log \left[e^{\frac{\lambda}{2}\sum E^{ij}_{kl}\sqrt{\mathbf{D}^i}\sqrt{\mathbf{D}^j}\frac{\partial}{\partial \tilde{Q}^i_k}\frac{\partial}{\partial \tilde{Q}^j_l}} \prod_m \tau_+(\lambda \mathbf{D}^m, \tilde{Q}^m) \right]_{\tilde{\mathbf{Q}}=\sqrt{\mathbf{D}}R^{-1}\Psi\lfloor J\mathbf{q}\rfloor} \tag{51}$$

are equal. The formula of the Proposition is obtained by from (51) by expanding the variable change

$$\tilde{\mathbf{Q}} = \sqrt{\mathbf{D}}R^{-1}\Psi\lfloor J\mathbf{q}\rfloor$$

removing the dilaton shift. $\square$

The descendent formula of Proposition 2 coincides with the definition of the higher genus potentials in Chapter 6 after the restriction $\mathbf{t} \to t$. The result is obtained by the specialization of

$$\left[-\left(\sum_{k,j} (-z)^{k-1} \langle\langle \tilde{\epsilon}_j, \mathbf{1}^{k+1}, \mathbf{t}(\psi) - \psi, \rangle\rangle \right) R^j_i(z) \right]_{t=t_{crit,p}(\mathbf{t}_p)}$$

to

$$\sum_j \frac{1}{\sqrt{\Delta^j}} R^j_i(z)$$

after the restriction.

4.2. ***M*-invariance revisited.** Let $\mathcal{F}$ be a complex Frobenius manifold of dimension m with R and J-calibrations defined on a open set U of a semisimple point p.

Let $p' \in U$. The variables sets $\mathbf{t}_{p'}$ agree for variables with positive descendent index. Let $\mathcal{D}$ be a finite differential operator in the variables $\{\mathbf{t}^i_k\}_{k>0}$. Let $g \geq 2$, and consider the function

$$\mathcal{D}\mathbf{F}^{\mathcal{F}}_{g,p'}|_{\mathbf{t}\to t}, \tag{52}$$

where the restriction takes place *after* the differentiation.

By Theorem 2 applied to the semisimple point p', the function (52) is *holomorphic* on U. Moreover, Theorem 2 also shows that the p'-dependence of the function (52) is holomorphic. By Proposition 3,

$$\frac{\partial}{\partial u^i} \log \tau^{\mathcal{F}} = \frac{1}{\tau^{\mathcal{F}}} \frac{\partial \tau^{\mathcal{F}}}{\partial u^i} = 0.$$

We conclude that the function (52) is independent of p'.

By Proposition 5, the same argument proves the M-invariance of the coefficients of the differential $d\mathbf{F}_1^{\mathcal{F}}$.

4.3. Givental's conjecture. Formal Frobenius manifolds $\mathcal{F}(X)$ obtained from Gromov-Witten theory carry natural higher genus descendent potentials defined by geometry:

$$F_g^X = \sum_{n,\beta} \frac{Q^\beta}{n!} \langle \gamma, \ldots, \gamma \rangle_{g,n,\beta}^X \tag{53}$$

where $\gamma = \sum_\mu t_k^\mu T_\mu \psi^k$. These potentials encode the higher genus descendent Gromov-Witten invariants of X.

We now assume $\mathcal{F}(X)$ is a complex semisimple Frobenius manifold near the origin. Since $\mathcal{F}(X)$ is conformal, a canonical R-calibration (up the signs of $\sqrt{\Delta^i}$) is defined. A canonical J-calibration is obtained from the Gromov-Witten theory.

We therefore have *two* definitions of the descendent potentials in genus $g \geq 1$. The first is the Givental potential obtained from the calibrated Frobenius manifold $\mathcal{F}(X)$. By the argument of Lemma 39 of Chapter 6, the Givental potential is *independent* of the square root choice in the conformal case. The second is the Gromov-Witten potential obtained from geometry (53).

Givental's Conjecture. *The Givental and Gromov-Witten potentials of $\mathcal{F}(X)$ are equal at every semisimple point of $\mathcal{F}(X)$.*

As in the case of primary fields, Givental's conjecture is proven in genus 1 but remains open in higher genus.

CHAPTER 11

Virasoro operators *(to be written)*

Bibliography

[1] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[2] R. Bott, *A residue formula for holomorphic vector fields*, JDG **4** (1967), 311–332.

[3] R. Dijkgraaf, E. Verlinde, and H. Verlinde, Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity, Nuclear Phys. B **348** (1991), 435–456.

[4] B. Dubrovin, *Geometry of* $2D$ *topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, LNM **1620**, Springer-Verlag:Berlin, 1996.

[5] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199.

[6] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry, Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, AMS: Providence, RI, 1997.

[7] A. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **13** (1996), 613–663.

[8] A. Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107–155, World Sci. Publishing: River Edge, NJ, 1998.

[9] A. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices **23** (2001), 1265–1286.

[10] A. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), 551–568.

[11] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. **168** (1995), 609–641.

[12] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.

[13] B. Iverson, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.

[14] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math. **129**, Birkhuser: Boston, 1995.

[15] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562.

[16] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models, I*, math.AG/0101147.

[17] Yu. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications **47**, AMS: Providence, RI, 1999.

[18] R. Pandharipande, *Rational curves on hypersurfaces (after A. Givental)*, Sminaire Bourbaki. Vol. 1997/98. Astrisque **252** (1998), 307–340.

[19] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, JDG **42** (1995), 259–367.

[20] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1** (1991), 243-310.