Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints

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Contents

Part 1. Frobenius manifolds and Givental's formula 5

Chapter 1. Frobenius manifolds 7
1. Overview 7
2. Frobenius manifolds 7
3. Semisimple Frobenius manifolds 10
4. Canonical coordinates and fundamental solutions 15

Chapter 2. Frobenius manifolds and Gromov-Witten theory 29
1. Overview 29
2. Frobenius manifolds over R 29
3. Formal Frobenius manifolds 30
4. Criteria for semisimplicity 31
5. Gromov-Witten theory 34
6. Equivariant Gromov-Witten theory 40

Chapter 3. Localization 43
1. $\mathbf{T}$-actions 43
2. Localization 43
3. $\mathbf{T}$-actions on projective spaces 44
4. $\mathbf{T}$-actions on $\overline{M}_{g,n}(\mathbf{P}^m, d)$ 45
5. Tautological classes 47
6. The Localization formula 47
7. Gravitational descendents 50

Chapter 4. Materialization 53
1. Overview 53
2. Semisimplicity of equivariant cohomology 53
3. Graph contributions 54
4. Derivative notation 55
5. Materialization 56
6. Local splitting equations 57
7. Materialization, norms, and the transition matrix 64

Chapter 5. Quantum potentials, the string flow, and the dilaton flow 69

1. Overview 69
2. Quantum potentials in genus 0 69
3. The string flow 72
4. Proof of Proposition 5 72
5. The dilaton flow 75
6. The string and dilaton flows 75

Chapter 6. Givental's formula for higher genus potentials 77
1. Genus 0 potentials 77
2. R-Calibrations 77
3. Genus 1 potentials 77
4. Higher genus potentials 78
5. The conformal case 80

Chapter 7. Givental's formula in equivariant Gromov–Witten theory 83
1. Overview 83
2. The functions A^i 83
3. The canonical calibration 84
4. The functions E^{ij}_{kl} for the canonical calibration 87
5. The functions T^i_k for the canonical R-calibration 90
6. Modified R-calibrations 92

Bibliography 95

Part 1

Frobenius manifolds and Givental's formula

In Part 1, Frobenius manifolds are introduced and several topics required for the statement of Givental's formula are presented. Semisimplicity, canonical coordinates, and fundamental solutions for Frobenius manifolds are covered in Chapter 1. In Chapter 2, Frobenius manifolds obtained from Gromov-Witten theory are discussed with a particular emphasis on the equivariant case. After a presentation of localization methods in Chapter 3, a complete development of Givental's materialization of canonical coordinates in equivariant Gromov-Witten theory is given in Chapter 4. The string and dilaton flows are treated in Chapter 5. Part 1 ends with the statement of Givental's formula for higher genus potentials for semisimple conformal Frobenius manifolds in Chapter 6. The formula for equivariant Gromov-Witten theory is presented in Chapter 7. We restrict the discussion of Givental's formula to the case of primary fields. The full descendent formula will be treated in Parts 2 and 3.

CHAPTER 1

Frobenius manifolds

1. Overview

Frobenius manifolds arise naturally: the quantum cohomology of a nonsingular projective variety X determines a formal Frobenius supermanifold over the Novikov ring of X. In Chapter 1, we will discuss complex Frobenius manifolds. Frobenius manifolds over the Novikov ring will be treated in Chapter 2.

If X has only even cohomology, the Frobenius structure determined by Gromov-Witten theory is *even* — we will restrict our attention here to the even case. The supertheory in the odd case is parallel except for the discussion of semisimplicity and canonical coordinates.

The main result of the Chapter 1 is Theorem 1 concerning flat vector fields for the Dubrovin connection in canonical coordinates. Theorem 1 plays an essential role in the subject.

2. Frobenius manifolds

2.1. Definitions. An (even) complex Frobenius manifold $\mathcal{F}$ consists of four mathematical structures $(M, g, A, \mathbf{1})$:

- M is a complex manifold of dimension m,
- g is a holomorphic, symmetric, non-degenerate quadratic form on the complex tangent bundle TM,
- A is a holomorphic symmetric tensor,

$$A : TM \otimes TM \otimes TM \to \mathcal{O}_M,$$

- $\mathbf{1}$ is a holomorphic vector field on M.

A and g together define a commutative product $*$ on TM by:

$$\langle X * Y, Z\rangle = A(X, Y, Z),$$

where X, Y, Z are holomorphic vector fields and $\langle , \rangle$ denotes the metric g. A *unit vector field* is a left and right identity for the $*$-product.

A *complex Frobenius manifold* $\mathcal{F}$ is a quadruple $(M, g, A, \mathbf{1})$ satisfying the following conditions:

(i) *Flatness*: g is a flat holomorphic metric.

(ii) *Potential*: M is covered by open sets U each equipped with a commuting basis of g-flat holomorphic vector fields,

$$X_1, \ldots, X_m \in \Gamma(U, TM),$$

and a holomorphic potential function $\Phi \in \Gamma(U, \mathcal{O}_U)$ such that

$$A(X_i, X_j, X_k) = X_i X_j X_k(\Phi),$$

(iii) *Associativity*: $*$ is an associative product.

(iv) *Unit*: $\mathbf{1}$ is a g-flat unit vector field.

The associativity condition (iii) is equivalent to the Witten-Dijkgraaf-Verlinde-Verlinde equations,

$$\langle (X_i * X_j) * X_k, X_l \rangle = \langle X_i * (X_j * X_k), X_l \rangle, \tag{1}$$

for all indices i, j, k, and l.

Let ∇ denote the holomorphic Levi-Civita connection obtained from the metric. For $z \in \mathbb{C}^*$, define the Dubrovin (projective) connection ∇_z by

$$\nabla_{z,X}(Y) = \nabla_X(Y) - \frac{1}{z} X * Y.$$

The WDVV equations (1) are also equivalent to the flatness of ∇_z for all $z \neq 0$.

The existence and flatness of the unit vector field V are not always required in the definition of a complex Frobenius manifold — see, for example, [**15**]. We will assume condition (iv) for the complex Frobenius manifolds considered here.

A C^∞-Frobenius manifold is defined by requiring all the structures $(M, g, A, \mathbf{1})$ to be defined in the C^∞-category.

2.2. Flat coordinates and fundamental solutions. Let $\mathcal{F}$ be a complex Frobenius manifold. Let p be a point of M. As g is flat, holomorphic flat coordinates $t^1, \ldots, t^m$ may be found in a neighborhood U of p. Let

$$\partial_i = \frac{\partial}{\partial t^i}$$

denote the corresponding flat vector fields. The convention,

$$\mathbf{1} = \partial_1$$

will usually be followed. We not will *not* assume $t^i(p) = 0$.

Let $g_{ij} = \langle \partial_i, \partial_j \rangle$, and let g^{ij} denote the inverse matrix. The metric functions g_{ij}, g^{ij} are constant on U (if U is connected).

A holomorphic vector field F on U may be written locally as $\sum_j f^j \partial_j$. The connection ∇_z is determined by:

$$\nabla_{z,i} F = \sum_j \frac{\partial f^j}{\partial t^i} \partial_j - \frac{1}{z} \partial_i * F,$$

where $\nabla_{z,i}$ denotes ∇_{z,∂_i}.

An $m \times m$ fundamental solution matrix $S_{ab}(z, t^1, \ldots, t^m)$ may be found near p for the differential equations defining ∇_z-flat vector fields: for all i,

$$\nabla_{z,i} \sum_{a,s} S_{ab}(z,t) g^{as} \partial_s = 0 \tag{2}$$

with the initial conditions

$$S_{ab}(z,p) = g_{ab}. \tag{3}$$

In fact, the matrix $S_{ab}(z,t)$ is uniquely defined by the flatness equations (2) and the initial conditions (3).

In case U is simply connected, the solution matrix S is well-defined on U with holomorphic coefficients $S_{ab}(z,t) \in \Gamma(\mathbb{C}^* \times U, \mathcal{O}_{\mathbb{C}^* \times U})$.

The initial conditions imply a unitary property at p:

$$\sum_{a,a'} S_{ab}(z,p) g^{aa'} S_{a'b'}(-z,p) = g_{bb'}$$

which is equivalent to

$$\langle \sum_{a,s} S_{ab}(z,p) g^{as} \partial_s, \sum_{a',s'} S_{a'b'}(-z,p) g^{a's'} \partial_{s'} \rangle_p = g_{bb'}.$$

A direct application of the Lemma below proves

$$\sum_{a,a'} S_{ab}(z,t) g^{aa'} S_{a'b'}(-z,t) = g_{bb'} \tag{4}$$

for all t near 0.

Lemma 1. *If W_+ and W_- are ∇_z-flat and ∇_{-z}-flat vector fields on M, then $\langle W_+, W_- \rangle$ is a locally constant function on M.*

Proof. Let X be any holomorphic vector field. Since ∇ is a metric connection, we find

$$X \langle W_+, W_- \rangle = \langle \nabla_X W_+, W_- \rangle + \langle W_+, \nabla_X W_- \rangle.$$

However, the flatness conditions imply

$$\nabla_X W_+ = \frac{1}{z} X * W_+, \quad \nabla_X W_- = -\frac{1}{z} X * W_-.$$

By the definition of $*$ and the symmetry of A, we conclude:

$$X\langle W_+, W_-\rangle = \frac{1}{z}\langle X * W_+, W_-\rangle - \frac{1}{z}\langle W_+, X * W_-\rangle = 0.$$

$\square$

2.3. Conformal Frobenius manifolds. Let $\mathcal{F} = (M, g, A, \mathbf{1})$ be a complex Frobenius manifold. Let E be a holomorphic vector field on M. Let $\mathcal{L}_E$ denote the Lie derivative. E is an *Euler vector field* on M if the following three conditions are satisfied:

(i) $\mathcal{L}_E(g) = (2 - D)g$ for a constant D,
(ii) $\mathcal{L}_E(*) = r\, *$ for a constant r,
(iii) $\mathcal{L}_E(\mathbf{1}) = v\mathbf{1}$ for a constant v.

The Lie derivatives in conditions (i) and (ii) may be written as

$$\mathcal{L}_E(g)(X, Y) = E(g(X, Y)) - g([E, X], Y) - g(X, [E, Y]), \tag{5}$$

$$X\ \mathcal{L}_E(*)\ Y = [E, X * Y] - [E, X] * Y - X * [E, Y].$$

A *conformal complex Frobenius manifold* is a complex Frobenius manifold together with an Euler vector field.

An Euler field is *normalized* if $r = 1$. As an arbitrary Euler field E can be normalized by scaling by the factor $\frac{1}{r}$, we will restrict our attention to normalized Euler fields.

3. Semisimple Frobenius manifolds

3.1. Characteristic varieties. Let $\mathcal{F} = (M, g, A, \mathbf{1})$ be a complex Frobenius manifold. Let $S^*(TM) = \oplus_i \mathrm{Sym}^i(TM)$ denote the symmetric algebra of the vector bundle TM. There is canonical surjection of sheaves of $\mathbb{C}$-algebras over M:

$$S^*(TM) \to TM \to 0, \tag{6}$$

where the algebra structure on TM is defined by the $*$-product. As $\mathrm{Spec}(S^*(TM))$ is canonically isomorphic to T^*M, a canonical embedding

$$\mathrm{Spec}(TM) \subset T^*M$$

is determined by sequence (6). $\mathrm{Spec}(TM)$ is the *characteristic subvariety* of T^*M determined by the Frobenius structure.

Let $p \in M$. The fiber of the characteristic subvariety over p is

$$\mathrm{Spec}(TM_p) \subset T^*M_p.$$

A subscheme of an affine space is *nondegenerate* if the subscheme does not lie in any linear hypersurface. The fiber $\mathrm{Spec}(TM_p)$ is a nondegenerate Artinian subscheme of length $m = dim_{\mathbb{C}}(M)$ of TM_p^*. The structure map of the characteristic variety,

$$\pi : \mathrm{Spec}(TM) \to M,$$

is finite and flat.

3.2. Semisimple points. An Artinian $\mathbb{C}$-algebra R is *semisimple* if there exists an algebra isomorphism

$$R \stackrel{\sim}{=} \oplus_1^{dim(R)} \mathbb{C},$$

where the direct sum algebra structure is taken on the right. A point $p \in M$ is *semisimple* if the tangent algebra $(TM_p, *)$ is a semisimple algebra. Equivalently, p is semisimple if the characteristic variety is étale over p.

Let $M_{ss} \subset M$ denote the set of semisimple points. The Frobenius manifold $\mathcal{F}$ is defined to be *semisimple* if $M_{ss} \subset M$ is dense. If M is connected and M_{ss} is non-empty, then $\mathcal{F}$ is semisimple.

3.3. Canonical coordinates. Let $M_{ss} \in M$ be the locus of semisimple points. Let $C_{ss} = \pi^{-1}(M_{ss})$ be the nonsingular open set of the characteristic subvariety lying over M_{ss}:

$$C_{ss} \subset T^*M_{ss}.$$

The cotangent space T^*M_{ss} is a holomorphic symplectic manifold with a canonical symplectic form ω. The proof of the following result is due to N. Reshetikhin [**10**].

Lemma 2. *C_{ss} is a Lagrangian subvariety of (T^*M_{ss}, ω).*

Proof. Let $p \in M_{ss}$. Let $t^1, \ldots, t^m$ be flat holomorphic coordinates near p (following the notation of Section 2.2). TM is trivialized near p by the vector fields ∂_i. Let

$$A_i : TM \to TM$$

denote the endomorphism defined near p by $*$-multiplication with ∂_i. A_i is an $m \times m$ matrix of holomorphic functions via the trivialization of TM,

$$\partial_i * \partial_a = \sum_b [A_i]_a^b \partial_b.$$

Let $\alpha = \sum_i A_i dt^i$ be a matrix valued 1-form. The constraints,

$$d\alpha = 0, \quad \alpha \wedge \alpha = 0,$$

are easily obtained from the potential and associativity conditions in the definition of $\mathcal{F}$. Here, the wedge product denotes matrix multiplication of matrix valued 1-forms.

As the characteristic subvariety is étale over p, holomorphic 1-forms

$$\gamma_1, \ldots, \gamma_m,$$

$$\gamma_j = \sum_i \gamma_{ji} dt^i$$

can be found such that $\mathrm{Spec}(TM) \subset T^*M$ is the union of the graphs of γ_j near p. To prove the characteristic subvariety is Lagrangian over a neighborhood of p, it suffices to prove all the 1-forms γ_j are closed.

Since the characteristic subvariety is nondegenerate, a basis of independent vector fields,

$$\epsilon_1, \ldots, \epsilon_m, \tag{7}$$

can be found near p satisfying:

$$\langle \epsilon_i, \gamma_j \rangle = \delta_{ij}, \tag{8}$$

where $\langle , \rangle$ denotes the canonical pairing between vector fields and 1-forms. Moreover, the vector fields ϵ_j and the 1-forms γ_j are in canonical bijective correspondence. In Section 3.4 below, the vector fields $\epsilon_1, \ldots, \epsilon_m$ will be seen to determine a basis of idempotents for the $*$-product.

By the construction of the characteristic subvariety, we find:

$$A_i \epsilon_j = \gamma_{ji} \epsilon_j.$$

The vector field basis $\epsilon_1, \ldots, \epsilon_m$ simultaneously diagonalizes the transformations A_i. The 1-forms γ_j are therefore in bijective correspondence with the simultaneous eigenspaces $\mathbb{C}\epsilon_j$ of the transformations $A_1, \ldots, A_m$. The eigenvalue for A_i of the eigenspace corresponding to γ_j is simply γ_{ji}. Since the vector fields $\epsilon_1, \ldots, \epsilon_m$ are independent, the simultaneous eigenspaces $\mathbb{C}\epsilon_1, \ldots, \mathbb{C}\epsilon_m$ are the complete set of eigenspaces (each of multiplicity 1).

We will view ϵ_j and γ_j as vectors of functions (via their expressions in the dual bases determined by ∂_i and dt^i respectively) when found inside the brackets $\langle , \rangle$ on the right side in the main calculation below. The $m \times m$ matrix of functions A_i acts on the vectors ϵ_j. Similarly, $d\epsilon_j$ and $d\gamma_j$ will be vectors of the corresponding 1-forms inside the brackets.

By definition, $\gamma_j = \sum_{i=1}^m \langle A_i \epsilon_j, \gamma_j \rangle dt^i$. The main calculation required for the Lemma is:

$$\begin{aligned}
d\gamma_j &= d\sum_{i=1}^m \langle A_i\epsilon_j, \gamma_j\rangle dt^i \\
&= \sum_{i=1}^m \langle (dA_i)\epsilon_j, \gamma_j\rangle dt^i + \langle A_i d\epsilon_j, \gamma_j\rangle dt^i + \langle A_i\epsilon_j, d\gamma_j\rangle dt^i \\
&= \sum_{i=1}^m \langle d\epsilon_j, A_i^\dagger \gamma_j\rangle dt^i + \langle A_i\epsilon_j, d\gamma_j\rangle dt^i \\
&= d\langle \epsilon_j, \gamma_j\rangle \wedge \gamma_j \\
&= 0.
\end{aligned}$$

The third equality uses the relation $d\alpha = 0$. In the third line, $A_i^\dagger$ denotes the adjoint of Q_i with respect to the canonical pairing (8). The fourth equality uses the relation

$$A_i^\dagger \gamma_j = \gamma_j^i \gamma_j.$$

The proof of the Lemma is complete. $\square$

The 1-forms $\gamma_1, \ldots, \gamma_m$ are uniquely specified (up to permutation). Since $d\gamma_j = 0$, we can find a holomorphic function u^j near p satisfying $du^j = \gamma_j$. The function u^j is uniquely specified (up to an integration constant). By the nondegeneracy property, the functions

$$u^1, \ldots, u^m$$

determine a *canonical coordinate system* of M near p.

There are two standard methods to specify constants in the construction of canonical coordinates. First, the constants may be fixed by requiring $u^j(p) = 0$ for all j. Second, in the conformal case, the Euler field will be shown to provide a canonical specification of the constants in Lemma 5 of Section 3.6. However, we will view the functions $\{u^j\}$ as canonical coordinates for *any* choice of integration constants.

3.4. Idempotents. Let p be a semisimple point of a complex Frobenius manifold $\mathcal{F}$. Let $u^1, \ldots, u^m$ be canonical coordinates defined on an open set U containing p. Define a basis of independent vector fields $\epsilon_1, \ldots, \epsilon_m$ on U by

$$\epsilon_i = \frac{\partial}{\partial u^i}.$$

LEMMA 3. *The vector fields ϵ_i are idempotents for the $*$-product:*

$$\epsilon_i * \epsilon_j = \delta_{ij}\epsilon_i.$$

Proof. In canonical coordinates, the characteristic subvariety is the union of the sections $du^1, \dots, du^m$ of T^*M. The Lemma is a direct consequence of this presentation of the characteristic subvariety. $\square$

3.5. Metric. In the next Lemma, the metric g is shown to be diagonal in the basis determined by the vector fields ϵ_i on U.

Lemma 4. *Let $p \in U$. The pairing $\langle \epsilon_i, \epsilon_j \rangle_p$ vanishes if and only if $i \neq j$.*

Proof. Using Lemma 3 and the definition of the $*$-product, we find:

$$\langle \epsilon_i, \epsilon_j \rangle_p = \langle \epsilon_i * \epsilon_i, \epsilon_j \rangle_p = \langle \epsilon_i, \epsilon_i * \epsilon_j \rangle_p.$$

As $\epsilon_i * \epsilon_j = 0$ if $i \neq j$, we find $\langle \epsilon_i, \epsilon_j \rangle_p = 0$ in this case. The non-vanishing of $\langle \epsilon_i, \epsilon_i \rangle_p$ then follows from the nondegeneracy of g. $\square$

3.6. Euler fields. If $\mathcal{F}$ is conformal, the Euler field takes a simple form in canonical coordinates at a semisimple point $p \in M$.

Lemma 5. *If E is an Euler field satisfying $\mathcal{L}_E(*) = r*$, then*

$$E = r \cdot \sum_i (u^i + c^i)\epsilon_i$$

in canonical coordinates (where $c^i \in \mathbb{C}$ are constants).

Proof. Let $E = \sum_i E^i(u)\epsilon_i$ for holomorphic functions $E^i(u)$. Then,

$$\begin{aligned} \epsilon_j \; \mathcal{L}_E(*) \; \epsilon_k &= [E, \epsilon_j * \epsilon_k] - [E, \epsilon_j] * \epsilon_k - \epsilon_j * [E, \epsilon_k] \\ &= -\delta_{jk} \sum_i \frac{\partial E^i}{\partial u^j}\epsilon_i + \frac{\partial E^k}{\partial u^j}\epsilon_k + \frac{\partial E^j}{\partial u^k}\epsilon_j. \end{aligned}$$

By the conformal property,

$$\epsilon_j \; \mathcal{L}_E(*) \; \epsilon_k = r \cdot (\epsilon_j * \epsilon_k) = r \cdot \delta_{jk}\epsilon_j.$$

Hence, $\partial E^i / \partial u^j = r \; \delta_{ij}$. $\square$

By Lemma 5, there is a unique choice of canonical coordinates (up to permutation) such that the Euler field takes a homogeneous form:

$$E = r \sum_i u^i \epsilon_i. \tag{9}$$

In the conformal semisimple case, we will always select canonical coordinates by requiring condition (9).

4. Canonical coordinates and fundamental solutions

4.1. Coordinate relationships. Let $\mathcal{F} = (M, g, A, \mathbf{1})$ be a complex Frobenius manifold and let $p \in M$ be a semisimple point. Let $U \subset M$ be an open set containing p where both flat coordinates and canonical coordinates are defined. In Section 4, we will use Greek indices exclusively for flat coordinates $\{t^\mu\}$ and Roman indices for canonical coordinates $\{u^i\}$. There is a tension between these coordinate choices for $\mathcal{F}$: the metric is trivial in flat coordinates and the $*$-product is trivial in canonical coordinates.

We follow the notation of Section 2.2 for flat coordinates $\{t^\mu\}$. The coordinate vector fields are $\{\partial_\mu\}$. The metric is specified by the symmetric matrix $g_{\mu\nu} = \langle \partial_\mu, \partial_\nu \rangle$. The inverse matrix is denoted by $g^{\mu\nu}$.

We follow the notation of Section 3.4 for canonical coordinates $\{u^i\}$. The coordinate vector fields are $\{\epsilon_i\}$. Let

$$\Delta^i = \langle \epsilon_i, \epsilon_i \rangle^{-1}.$$

We assume the existence of square roots $\sqrt{\Delta^i}$ of Δ_i near p. If U is simply connected, there exist holomorphic square roots of Δ^i since the functions Δ^i do not vanish. The square roots $\sqrt{\Delta^i}$ are unique up to sign.

Define normalized vector fields $\tilde{\epsilon}_i$ by a rescaling:

$$\tilde{\epsilon}_i = \sqrt{\Delta^i}\epsilon_i. \tag{10}$$

The metric pairing is

$$\langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle = \delta_{ij},$$

where δ_{ij} denotes the Kronecker symbol. The vector fields $\tilde{\epsilon}_i$ are orthonormal.

4.2. The transition matrix. Let Ψ be the transition matrix between the bases ∂_μ and $\tilde{\epsilon}_i$ of vector fields:

$$\sum_\mu a^\mu \partial_\mu = \sum_i \sum_\mu \Psi^i_\mu a^\mu \tilde{\epsilon}_i,$$

$$\sum_i b^i \tilde{\epsilon}_i = \sum_\mu \sum_i (\Psi^{-1})^\mu_i b^i \partial_\mu.$$

By the orthonormality of $\tilde{\epsilon}_i$, the elements of Ψ are:

$$\Psi^i_\mu = \langle \tilde{\epsilon}_i, \partial_\mu \rangle.$$

By convention, the upper index denotes the row and the lower index denotes the column.

Lemma 6. *The matrix Ψ satisfies:*

(i) $\sum_i \Psi^i_\mu \Psi^i_\nu = g_{\mu\nu}$,
(ii) $\sum_{\mu,\nu} \Psi^i_\mu g^{\mu\nu} \Psi^j_\nu = \delta^{ij}$.

Proof. Both properties follow easily from the definitions:

$$\begin{aligned}
\sum_i \Psi^i_\mu \Psi^i_\nu &= \sum_{i,j} \Psi^i_\mu \Psi^j_\nu \langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle \\
&= \langle \sum_i \Psi^i_\mu \tilde{\epsilon}_i, \sum_j \Psi^j_\nu \tilde{\epsilon}_j \rangle \\
&= \langle \partial_\mu, \partial_\nu \rangle \\
&= g_{\mu\nu},
\end{aligned}$$

$$\begin{aligned}
\sum_{\mu,\nu} \Psi^i_\mu g^{\mu\nu} \Psi^j_\nu &= \sum_{\mu,\nu} \langle \tilde{\epsilon}_i, \partial_\mu \rangle g^{\mu\nu} \langle \partial_\nu, \tilde{\epsilon}_j \rangle \\
&= \langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle \\
&= \delta^{ij}.
\end{aligned}$$

□

Denote the transpose of a matrix M by M^t. M is *skew symmetric* if $M + M^t = 0$.

LEMMA 7. $\Psi d\Psi^{-1}$ *is a skew symmetric matrix of 1-forms.*

Proof. Lemma 6 part (ii) may be written in matrix from:

$$\Psi g^{-1} \Psi^t = 1, \tag{11}$$

where 1 is the identity matrix. After applying d, we obtain:

$$d\Psi g^{-1} \Psi^t + \Psi g^{-1} d\Psi^t = 0, \tag{12}$$

since g has constant coefficients. From (11), we also obtain

$$\begin{aligned}
\Psi^{-1} &= g^{-1} \Psi^t, \\
d\Psi^{-1} &= g^{-1} d\Psi^t, \\
\Psi d\Psi^{-1} &= \Psi g^{-1} d\Psi^t. \qquad (13)
\end{aligned}$$

Since g is symmetric,

$$(d\Psi^{-1})^t = d\Psi g^{-1}.$$

Hence,

$$(\Psi d\Psi^{-1})^t = d\Psi g^{-1} \Psi^t \tag{14}$$

By combining equations (12)- (14), we obtain:

$$\Psi d\Psi^{-1} + (\Psi d\Psi^{-1})^t = 0, \tag{15}$$

concluding the proof of the Lemma. $\square$

Since the diagonal entries of skew symmetric matrices vanish, we obtain the following result.

LEMMA 8. $\Psi d\Psi^{-1}$ *vanishes along the diagonal.*

4.3. Connections in flat and canonical coordinates. Additional properties of the transition matrix Ψ can be found by studying the Levi-Civita and Dubrovin connections. We may view these connections as maps:

$$\nabla, \nabla_z : \Gamma(TM) \to \Gamma(TM \otimes TM^*).$$

Once a trivialization of TM is selected, the connections define maps:

$$\nabla, \nabla_z : \mathbb{C}^m \otimes \mathcal{O}_M \to \mathbb{C}^m \otimes \Gamma(TM^*).$$

In the $\{\partial_\mu\}$ basis, the Levi-Civita connection is identified with the differential,

$$\nabla = d,$$

and the Dubrovin connection,

$$\nabla_z = d - \frac{1}{z}\alpha,$$

is determined by the matrix of 1-forms α obtained from the $*$-product (see Section 3.3).

In the $\{\tilde{\epsilon}_i\}$ basis, the Levi-Civita connection takes the form:

$$\nabla = d + \Psi d\Psi^{-1}.$$

By the flatness of the metric, $\nabla^2 = 0$.

LEMMA 9. $d\Psi \wedge d\Psi^{-1} + \Psi d\Psi^{-1} \wedge \Psi d\Psi^{-1} = 0.$

Proof. Expanding the operator ∇^2, we find

$$\begin{aligned} \nabla^2 &= (d + \Psi d\Psi^{-1})^2 \\ &= d\Psi \wedge d\Psi^{-1} + \Psi d\Psi^{-1} \wedge \Psi d\Psi^{-1}. \end{aligned}$$

As $\nabla^2 = 0$, the Lemma follows. $\square$

In the $\{\tilde{\epsilon}_i\}$ basis, the Dubrovin connection may be written as:

$$\nabla_z = \nabla - \frac{1}{z} d\mathbf{u}, \tag{16}$$

where $\mathbf{u}$ is the diagonal matrix of functions,

$$\mathbf{u} = \mathrm{Diag}(u^1, \ldots, u^m).$$

By the flatness of the Dubrovin connection, $\nabla_z^2 = 0$.

Lemma 10. *For all indices i and j:*

$$(\Psi d\Psi^{-1})^i_j \wedge (du^j - du^i) = 0.$$

Proof. We may calculate ∇_z^2 by expanding (16) to obtain:

$$\nabla_z^2 = \nabla^2 - \frac{1}{z}\Psi d\Psi^{-1} \wedge d\mathbf{u} - \frac{1}{z}d\mathbf{u} \wedge \Psi d\Psi^{-1} = 0.$$

The resulting equation,

$$\Psi d\Psi^{-1} \wedge d\mathbf{u} + d\mathbf{u} \wedge \Psi d\Psi^{-1} = 0,$$

yields the Lemma. □

4.4. The conformal case. If $\mathcal{F}$ is conformal with Euler field E, we may consider the Lie derivative, $\mathcal{L}_E(\Psi d\Psi^{-1})$, defined by the Lie derivatives of the coefficient 1-forms.

Lemma 11. *In the conformal case, $\mathcal{L}_E(\Psi d\Psi^{-1}) = 0$.*

Proof. We may assume E to be normalized. Let $\{u^i\}$ be canonical coordinates defined by the conformal structure:

$$E = \sum_i u^i \epsilon_i.$$

Let $\mathcal{L}_E(g) = Dg$. We first prove the functions $(\Delta^i)^{-1}$ are homogeneous of equal degree.

$$\begin{aligned} \mathcal{L}_E((\Delta^i)^{-1}) &= \mathcal{L}_E(g(\epsilon_i, \epsilon_i)) \\ &= \mathcal{L}_E(g)(\epsilon_i, \epsilon_i) + g(\mathcal{L}_E(\epsilon_i), \epsilon_i) + g(\epsilon_i, \mathcal{L}_E(\epsilon_i)) \\ &= (D-2)(\Delta^i)^{-1}. \end{aligned}$$

Equivalently, the functions Δ^i are homogeneous of degree $2 - D$.

By the definition of the Levi-Civita connection in the coordinates $\{u^i\}$, the Christoffel symbols are:

$$\Gamma^i_{kj} = \frac{1}{2}(\delta_{ij}\frac{\partial(\Delta^j)^{-1}}{\partial u^k} + \delta_{ik}\frac{\partial(\Delta^k)^{-1}}{\partial u^i} - \delta_{kj}\frac{\partial(\Delta^j)^{-1}}{\partial u^i})\Delta^i.$$

As the functions Δ^i are homogeneous, we find $\mathcal{L}_E(\Gamma^i_{kj}) = -\Gamma^i_{kj}$.

In the basis defined by $\{\epsilon_i\}$, the Levi-Civita connection takes the form:

$$\nabla = d + \Gamma,$$

where Γ is a matrix of 1-forms with coefficients

$$\Gamma^i_j = \sum_k \Gamma^i_{kj} du^k.$$

Since $\mathcal{L}_E(du^k) = du^k$, we find $\mathcal{L}_E(\Gamma) = 0$.

Let $\sqrt{\mathbf{\Delta}}$ denote the diagonal matrix of functions:

$$\sqrt{\mathbf{\Delta}} = \mathrm{Diag}(\sqrt{\Delta^1}, \ldots, \sqrt{\Delta^m}).$$

Since the functions Δ^i are homogeneous of equal degree, the functions $\sqrt{\Delta^i}$ must also be homogeneous of equal degree. The transition matrix from the $\{\epsilon_i\}$ basis to the $\{\tilde{\epsilon}_i\}$ basis is $\sqrt{\mathbf{\Delta}}$. The Levi-Civita connection in the $\{\tilde{\epsilon}_i\}$ basis is:

$$\nabla = d + \sqrt{\mathbf{\Delta}}\; d\sqrt{\mathbf{\Delta}}^{-1} + \sqrt{\mathbf{\Delta}}\;\Gamma\;\sqrt{\mathbf{\Delta}}^{-1}.$$

We conclude:

$$\Psi d\Psi^{-1} = \sqrt{\mathbf{\Delta}}\; d\sqrt{\mathbf{\Delta}}^{-1} + \sqrt{\mathbf{\Delta}}\;\Gamma\;\sqrt{\mathbf{\Delta}}^{-1}.$$

As $\sqrt{\mathbf{\Delta}}$ is homogeneous and $\mathcal{L}_E(\Gamma) = 0$, the Lemma follows immediately. $\square$

A semisimple Frobenius manifold is *trivial* at $p \in M$ if the functions Δ^i are constant.

LEMMA 12. *Let $\mathcal{F}$ be conformal. Let $p \in M$ be a semisimple point at which the Euler field vanishes. Then, $\mathcal{F}$ is trivial at p.*

Proof. Since the Euler field vanishes at p, the canonical coordinates specified by the conformal structure satisfy $u^j(p) = 0$ for all j. Since Δ^i is a holomorphic function at p, Δ^i may be expanded in power series in $\{u^j\}$. Δ^i is an eigenfunction for E if and only if Δ^i is a homogeneous polynomial in the variables $\{u^j\}$. As Δ^i does not vanish at p, Δ^i must be a constant. $\square$

4.5. Fundamental solutions. The most technical part of the theory of semisimple Frobenius manifolds which will be needed for Givental's study of higher genus structures is the study of ∇_z-flat vector fields in canonical coordinates.

Let $\mathcal{F} = (M, g, A, \mathbf{1})$ be a semisimple complex Frobenius manifold. Let $p \in M$ be a semisimple point. Let U be a simply connected neighborhood of p carrying both flat $\{t^\mu\}$ and canonical $\{u^i\}$ coordinates.

We have already studied the differential equation for ∇_z-flat vector fields in flat coordinates. Following the notation of Section 2.2, the equation in flat coordinates,

$$\nabla_{z,\mu} \sum_{\alpha,\gamma} S_{\alpha\beta}(z,t) g^{\alpha\gamma} \partial_\gamma = 0,$$

with the initial conditions

$$S_{\alpha\beta}(z,p) = g_{\alpha\beta},$$

has a unique solution matrix $S_{\alpha\beta}(z,t)$ with coefficient functions in $\Gamma(\mathbb{C}^* \times U)$. Let

$$S^\gamma_\beta = \sum_\alpha S_{\alpha\beta} g^{\alpha\gamma},$$

then a basis of ∇_z-flat vector fields is given by $\sum_\gamma S^\gamma_\beta \partial_\gamma$ for $1 \le \beta \le m$.

We will now study the ∇_z-flat vector fields in canonical coordinates. A matrix of functions $\tilde{S}^k_j(z,u)$ is a fundamental solution in canonical coordinates if the vector fields

$$\sum_k \tilde{S}^k_j \tilde{\epsilon}_k$$

for $1 \le j \le m$ determine a basis of ∇_z-flat fields on U. Of course, $\tilde{S}^k_j$ is a fundamental solution in canonical coordinates if and only if

$$S^\gamma_j = (\Psi^{-1})^\gamma_k \tilde{S}^k_j$$

is a fundamental solution in flat coordinates.

THEOREM 1. *The differential equation in canonical coordinates for ∇_z-flat fields has the following properties:*

(i) *Formal fundamental solutions $\tilde{S}^k_j$ may be found in the form*

$$\tilde{S}(z,u) = R(z,u)e^{\mathbf{u}/z},$$

where $R(z,u)$ is an $m \times m$ matrix series in non-negative powers of z:

$$R(z,u) = \sum_{n=0}^{\infty} R_n(u)\ z^n, \quad R_0 = 1,$$

and $R_n(u)$ is a matrix of holomorphic functions on U.

(ii) *The matrix series $R(z,u)$ in (i) can be chosen to satisfy the* unitary *condition:*

$$R(z,u)R^t(-z,u) = 1.$$

(iii) *If $R(z,u)$ determines a fundamental solution in (i) and satisfies the unitary condition, then $R(z,u)$ is unique up to a right multiplication by a constant (in u) matrix*

$$\exp(\sum_{k\ge 1} \mathbf{a}_{2k-1} z^{2k-1})$$

where

$$\{\ \mathbf{a}_{2k-1} = Diag(a^1_{1,2k-1}, a^2_{2,2k-1}, \cdots, a^m_{m,2k-1})\ \}$$

are constant diagonal matrices.

(iv) *If $\mathcal{F}$ is a conformal complex Frobenius manifold, then there exists a unique matrix series $R(z,u)$ which determines a fundamental solution in (i) and is homogeneous (with $deg(z)=1$) with respect to the conformal structure. The unique homogeneous choice $R(z,u)$ is unitary.*

Parts (i-iii) of Theorem 1 are valid for *any* choice of canonical coordinates. For Parts (i-iii), if the canonical coordinates $\{u^i\}$ are chosen to satisfy $u^i(p)=0$, then the matrix coefficients of the formal fundamental solution $\tilde{S}(z,u)$ are Laurent series,

$$\tilde{S}^k_j(z,u) = \sum_{n\in\mathbb{Z}} C_n(u)\ z^n, \tag{17}$$

where $C_n(u) \in \mathbb{C}[[u^1,\ldots,u^m]]$. The Laurent property (17) is obtained by expanding the matrix functions $R_n(u)$ at p in power series in $\{u^i\}$ and then analyzing the product $R(z,u)e^{\mathbf{u}/z}$.

For Part (iv), the canonical coordinates specified by the conformal structure are required. In this case, $u^i(p)\neq 0$. The product $R(z,u)e^{\mathbf{u}/z}$ then can not be expanded in the form (17). The product $R(z,u)e^{\mathbf{u}/z}$ *must* be viewed a formal object. Nevertheless, the matrix series $R(z,u)$ is well-defined.

An *R-calibration* of a semisimple Frobenius manifold $\mathcal{F}$ at p is a selection of square roots $\sqrt{\Delta^i}$ together with a formal fundamental solution $\tilde{S}^k_j$ satisfying (i) and (ii) of Theorem 1. For fixed $\sqrt{\Delta^i}$, a single R-calibration determines all the other R-calibrations by (iii).

4.6. Proof of Theorem 1. Theorem 1 is proven by an explicit construction of formal fundamental solutions.

Part (i). $\tilde{S}^k_j$ is a fundamental solution in canonical coordinates if and only if

$$S^\gamma_j = (\Psi^{-1})^\gamma_k \tilde{S}^k_j$$

is a fundamental solution in flat coordinates, or equivalently,

$$(zd-\alpha)\Psi^{-1}\tilde{S} = 0.$$

The substitution $\tilde{S} = Re^{\mathbf{u}/z}$ then yields:

$$zd\Psi^{-1}Re^{\mathbf{u}/z} + z\Psi^{-1}dRe^{\mathbf{u}/z} + \Psi^{-1}Re^{\mathbf{u}/z}d\mathbf{u} - \alpha\Psi^{-1}Re^{\mathbf{u}/z} = 0.$$

After multiplying by $e^{-\mathbf{u}/z}$ on the right, we find the main flatness equation:

$$zd\Psi^{-1}R + z\Psi^{-1}dR + \Psi^{-1}Rd\mathbf{u} - \alpha\Psi^{-1}R = 0.$$

We will construct matrix coefficients $(R_n)_i^j$ of the series

$$R(z,u) = \sum_{n=0}^{\infty} R_n(u)\ z^n$$

satisfying the flatness equation inductively in n.

After expanding the flatness equation in powers of z, we find:

$$\Psi^{-1} R_0 d\mathbf{u} - \alpha \Psi^{-1} R_0 = 0 \tag{18}$$

in degree 0 and

$$d\Psi^{-1} R_{k-1} + \Psi^{-1} dR_{k-1} + \Psi^{-1} R_k d\mathbf{u} - \alpha \Psi^{-1} R_k = 0 \tag{19}$$

for degrees $k \geq 1$.

If $R_0 = 1$, equation (18) specializes to the relation:

$$d\mathbf{u} = \Psi \alpha \Psi^{-1}. \tag{20}$$

Since the vector fields ϵ_i are idempotents for the $*$-product, equation (20) is true. Hence, the matrix $R_0 = 1$ guarantees $\tilde{S} = Re^{\mathbf{u}/z}$ satisfies the flatness equation in degree 0 in z.

Equation (19) for $k = 1$ may be rewritten (using $d\mathbf{u} = \Psi\alpha\Psi^{-1}$) as:

$$\Psi d\Psi^{-1} = [d\mathbf{u}, R_1]. \tag{21}$$

To construct R_1, we first write equation (21) explicitly:

$$(\Psi d\Psi^{-1})_i^j = (du^j - du^i)(R_1)_i^j. \tag{22}$$

By Lemma 10, equation (22) can be solved uniquely to determine the off-diagonal coefficients of R_1. By Lemma 8, $\Psi d\Psi^{-1}$ vanishes on the diagonal. The right side of (22) trivially vanishes on the diagonal for *any* matrix R_1. The matrix R_1 (with as yet unspecified diagonal entries) guarantees $\tilde{S} = Re^{\mathbf{u}/z}$ satisfies the flatness equation in degree 1 in z.

For $k \geq 2$, equation (19) takes the equivalent form:

$$\nabla R_{k-1} = [d\mathbf{u}, R_k]. \tag{23}$$

As $d\mathbf{u}$ is a diagonal matrix, the diagonal coefficients of $[d\mathbf{u}, R_k]$ must vanish. Therefore, the diagonal coefficients of R_1 are constrained by (23) for $k = 2$:

$$(dR_1)_i^i + \sum_k (\Psi d\Psi^{-1})_k^i (R_1)_i^k = 0.$$

Using, equation (22), we find:

$$(dR_1)_i^i + \sum_k (du^i - du^k)(R_1)_k^i (R_1)_i^k = 0. \tag{24}$$

The second term of (24) is determined by the off-diagonal coefficients of R_1. Therefore, if (24) admits a solution, the diagonal coefficients of R_1 are determined up to additive constants.

To prove that equation (24) is solvable, we must show the matrix of 2-forms,

$$d(\Psi d\Psi^{-1} R_1),$$

determined by the off-diagonal coefficients of R_1, vanishes along the diagonal. The first step is a calculation:

$$\begin{aligned} d(\Psi d\Psi^{-1} R_1) &= d\Psi \wedge d\Psi^{-1} R_1 - \Psi d\Psi^{-1} \wedge dR_1 \\ &= -\Psi d\Psi^{-1} \wedge (dR_1 + \Psi d\Psi^{-1} R_1) \\ &= -\Psi d\Psi^{-1} \wedge \nabla R_1. \end{aligned}$$

Lemma 9 is used in the second equality above. The computation also shows the matrix $-\Psi d\Psi^{-1} \wedge \nabla R_1$ does *not* depend upon the diagonal coefficients of R_1.

It remains to show that $-\Psi d\Psi^{-1} \wedge \nabla R_1$ vanishes along the diagonal. Another result is required for this conclusion:

$$(du^j - du^i) \wedge (\nabla R_1)^j_i = 0. \tag{25}$$

Equation (25) is obtained from the following computation:

$$\begin{aligned} (du^j - du^i) \wedge (\nabla R_1)^j_i &= (d\mathbf{u} \wedge \nabla R_1 + \nabla R_1 \wedge d\mathbf{u})^j_i \\ &= -(\nabla [d\mathbf{u}, R_1])^j_i \\ &= -(\nabla (\Psi d\Psi^{-1}))^j_i \\ &= 0. \end{aligned}$$

The second and third equalities both use (21). The fourth equality follows from Lemma 9. The second equality of the computation shows that equation (25) does *not* depend upon the diagonal coefficients of R_1.

As Lemma 10 and equation (25) together imply the matrix

$$-\Psi d\Psi \wedge \nabla R_1$$

vanishes along the diagonal, the proof of the solvability of (24) is complete. R_1 is therefore well-defined up to additive constants on the diagonals.

For the construction of R_n for $n \geq 2$, we start with the following inductive assumptions at level n:

- The matrices R_k are determined for $k < n$ and equations (18-19) are satisfied for $k < n$.
- $(du^j - du^i) \wedge (\nabla R_{n-1})^j_i = 0$.
- $(\nabla R_{n-1})^i_i = 0$.

We will then construct a matrix R_n such that inductive assumptions are satisfied at level $n+1$.

The constructions of R_0 and R_1 show the inductive assumption are satisfied at level $n=2$. The induction procedure then completes the proof of Part (i) of Theorem 1.

To begin the construction of R_n, we first observe that equation (23) is solvable for $k=n$ precisely by the second and third assumptions at level n. The off-diagonal coefficients of R_n are uniquely determined by (23) for $k=n$.

The next step is to obtain the equation

$$(du^j - du^i)\wedge(\nabla R_n)^j_i = 0 \tag{26}$$

from the computation:

$$\begin{aligned}(du^j - du^i)\wedge(\nabla R_n)^j_i &= (d\mathbf{u}\wedge\nabla R_n + \nabla R_n\wedge d\mathbf{u})^j_i\\ &= -(\nabla[d\mathbf{u}, R_n])^j_i\\ &= -(\nabla^2 R_{n-1})^j_i\\ &= 0.\end{aligned}$$

The second equality shows (26) does *not* depend upon the diagonal coefficients of R_n.

The diagonal coefficients of R_n are constructed by using the equation $(\nabla R_n)^i_i = 0$, or equivalently,:

$$(dR_n)^i_i - \sum_{k\neq i}(\Psi d\Psi^{-1})^i_k (R_n)^k_i = 0. \tag{27}$$

The second term of (27) is determined by the off-diagonal coefficients of R_n. Therefore, if (24) admits a solution, the diagonal coefficients of R_n are determined up to additive constants.

To prove that equation (24) is solvable, we must show that the matrix of 2-forms,

$$d(\Psi d\Psi^{-1}R_n),$$

determined by the off-diagonal coefficients of R_n, vanishes along the diagonal. We start with a computation:

$$\begin{aligned}d(\Psi d\Psi^{-1}R_n) &= d\Psi\wedge d\Psi^{-1}R_n - \Psi d\Psi^{-1}\wedge dR_n\\ &= -\Psi d\Psi^{-1}\wedge\nabla R_n.\end{aligned}$$

The matrix $-\Psi d\Psi^{-1}\wedge\nabla R_n$ vanishes along the diagonal by Lemma 10 and equation (26). Therefore, equation (27) is solvable.

We have constructed R_n and simultaneously verified the inductive assumptions at level $n+1$. The proof of Part (i) of Theorem 1 is complete. $\square$

Part (ii). We have seen that $R(z,u)$ is not uniquely determined by the flatness equations: integration constants for the diagonal coefficients of R_n are unconstrained in the induction step of the construction for $n \geq 1$. We will show that the unitary condition,

$$R(z,u)R^t(-z,u) = 1,$$

can be achieved by an appropriate selection of these constants.

Define the matrix series $P(z,u) = \sum_{n \geq 0} z^n P_n(u)$ by:

$$P(z,u) = R(z,u)R^t(-z,u).$$

Since $R_0 = 1$, we find $P_0 = 1$. The unitary condition is equivalent to $P_n = 0$ for $n \geq 1$.

An adjustment result in the inductive construction of $R(z,u)$ is required for the unitary construction. Assume the matrices R_k have been constructed for $k \leq n$ by the procedure of the proof of Part (i). Assume P_n is a scalar matrix. Let R_{n+1} be the next matrix determined by the inductive construction of part (i) — well-defined up to integration constants along the diagonal. We will prove that by adjusting the integration constants of R_{n+1}, the matrix P_{n+1} can be forced to vanish. Application of the adjustment result at each stage in the construction process of part (i) yields a matrix series $R(z,u)$ satisfying the unitary condition.

The proof of the adjustment result starts with a Lemma also needed in the proof of Part (iv).

Lemma 13. *If P_n is a scalar, then P_{n+1} is diagonal with constant entries.*

Proof. A direct computation using (21-23) yields:

$$[d\mathbf{u}, P_{k+1}] = dP_k + [\Psi d\Psi^{-1}, P_k], \tag{28}$$

for $k \geq 0$. Since P_n is a scalar, equation (28) for $k = n$ implies the off-diagonal coefficients of P_{n+1} are zero.

There are now two cases. If $n+1$ is odd, then

$$(P_{n+1})^t = -P_{n+1}.$$

Therefore, the diagonal coefficients $(P_{n+1})_{ii}$ vanish and $P_{n+1} = 0$.

If $n+1$ is even, we will use equation (28) for $k = n+1$:

$$[d\mathbf{u}, P_{n+2}] = dP_{n+1} + [\Psi d\Psi^{-1}, P_{n+1}].$$

The left side above vanishes along the diagonal. The second term on the right side has no diagonal entries as $\Psi d\Psi^{-1}$ vanishes along the diagonal and P_{n+1} is diagonal. Therefore, $(dP_{n+1})_{ii} = 0$, or equivalently, P_{n+1} is diagonal with constant entires. $\square$

Hence, for the adjustment result, only the $n+1$ even case need be considered. If $n+1$ is even, we find

$$P_{n+1} = R_{n+1} + R_{n+1}^t + \sum_{j=1}^{n} (-1)^j R_{n+1-j} R_j^t.$$

Therefore, P_{n+1} can be forced to vanish along the diagonal by adjusting the integration constants along the diagonal of R_{n+1}. In fact, there is a *unique* choice of the diagonal constants to make P_{n+1} vanish. The proof of the adjustment result is therefore complete. $\square$

Part (iii). The proofs of Parts (i) and (ii) show the integration constants along the diagonals of R_n for odd n are unconstrained in the unitary construction. If $R(z,u)$ is a matrix series satisfying the unitary condition and $\mathbf{a}_{2k-1}$ are arbitrary diagonal matrices for $k \geq 1$, then the product series

$$R^{new}(z,u) = R(z,u) \exp(\sum_{k \geq 1} \mathbf{a}_{2k-1} z^{2k-1})$$

is easily seen to define a formal fundamental solution by $R^{new} e^{\mathbf{u}/z}$ and to satisfy the unitary condition.

Moreover, the matrices $\{\mathbf{a}_{2k-1}\}$ uniquely capture the freedom of the unconstrained diagonal integration constants in the construction of $R(z,u)$. The bijective correspondence between the matrices $\{\mathbf{a}_{2k-1}\}$ and the integration constants is proven by induction in k. $\square$

Part (iv). In the conformal case, let $E = \sum_i u^i \epsilon_i$ be the normalized Euler vector field in canonical coordinates near the semisimple point p. As $R_0 = 1$, we have

$$\mathcal{L}_E(R_0) = 0.$$

We will prove that the condition

$$\mathcal{L}_E(R_n) = -nR_n \tag{29}$$

can be satisfied at each stage in the inductive construction of $R(z,u)$. Moreover, the resulting homogeneous solution $R(z,u)$ is unique and unitary.

As $\mathcal{L}_E(\Psi d\Psi^{-1}) = 0$ by Lemma 11, the off-diagonal coefficients of R_1 are of degree -1 with respect to E by equation (22). The diagonal coefficients of R_1 can be *chosen* to be of degree -1 by equation (24) and the following Lemma.

LEMMA 14. *Let $k \neq 0$ be a constant. Let f be a holomorphic function on U satisfying $\mathcal{L}_E(df) = k \cdot df$. Then, $f + c$ is homogeneous of degree k for a unique constant c:*

Proof. Since $d\mathcal{L}_E(f) = \mathcal{L}_E(df) = k \cdot df$, we find:

$$d(\mathcal{L}_E(f) - k \cdot f) = 0.$$

As U is simply connected (and $k \neq 0$), the equation

$$\mathcal{L}_E(f) = k \cdot (f + c)$$

holds for a unique constant c. Then, $\mathcal{L}_E(f+c) = \mathcal{L}_E(f) = k(f+c)$. ☐

For the induction step, assume R_n is homogeneous of degree $-n$. Equation (23) then forces the off-diagonal coefficients of R_{n+1} to be homogeneous of degree $-(n+1)$. The diagonal coefficients of R_{n+1} can be chosen to be of degree $-(n+1)$ by equation (27) and Lemma 14.

Since the integration constants along the diagonal are fixed in the inductive construction by Lemma 14, the homogeneous series $R(z, u)$ is uniquely determined.

To prove the unitary condition for the homogeneous series $R(z, u)$, we first observe that the condition

$$\mathcal{L}_E(P_n) = -nP_n \tag{30}$$

is a direct consequence of (29). Certainly $P_0 = 1$. If P_n is a scalar, then P_{n+1} is diagonal with constant coefficients by Lemma 13. Then, $P_{n+1} = 0$ by equation (30). By induction, we conclude the unitary condition: $P_n = 0$ for $n \geq 1$. ☐

Corollary 1. $(d + \Psi d\Psi^{-1})R = [(\frac{d\mathbf{u}}{z}), R]$.

Proof. The Corollary is obtained immediately from equations (19) and (20) of the proof of Theorem 1. ☐

4.7. The endomorphism R. The interpretation of the matrix series R of Theorem 1 as an endomorphism series will play an important role. Given a formal fundamental solution,

$$\tilde{S} = R\, e^{\mathbf{u}/z},$$

define an endomorphism series in z,

$$R(z, u) : TM \to TM,$$

by the equation:

$$\langle \tilde{\epsilon}_j, R\tilde{\epsilon}_i \rangle = R_i^j.$$

The initial term $R(0, u)$ is the identity endomorphism. The unitary condition (ii) of Theorem 1 may be written as:

$$R(z, u)R^\dagger(-z, u) = 1,$$

where the adjoint is taken with respect to the metric. The endomorphism series is more natural than the matrix series.

Let $\mathcal{F}$ be a conformal Frobenius manifold and let p be a semisimple point. Once the square roots $\sqrt{\Delta^i}$ have been selected, there is a canonical homogeneous formal solution by part (iv) of Theorem 1.

LEMMA 15. *In the conformal case, the endomorphism series*

$$R : TM \to TM$$

obtained from the homogeneous formal solution is independent of the selection of the square roots $\sqrt{\Delta^i}$.

Proof. Let $\sqrt{\Delta^i}$ and $\sqrt{\Delta^{i'}}$ be two choices of square roots. Let D be the diagonal matrix with coefficients $\sqrt{\Delta^i}/\sqrt{\Delta^{i'}}$. If R is the unique homogeneous matrix series solution for $\sqrt{\Delta^i}$, then a simple verification shows that

$$R' = DRD$$

is the unique homogeneous matrix series solution for $\sqrt{\Delta^{i'}}$. $\square$

Therefore, in the conformal case, the endomorphism series

$$R : TM \to TM$$

at p is absolutely canonical.

CHAPTER 2

Frobenius manifolds and Gromov-Witten theory

1. Overview

Let X be a nonsingular complex projective variety. The Frobenius structures determined by quantum cohomology are defined over the Novikov ring of X and are formal. After a treatment of Frobenius manifolds over arbitrary base rings and a discussion of formal Frobenius manifolds, the Frobenius structures arising in Gromov-Witten theory will be introduced. If X is equipped with a torus action, richer Frobenius structures are determined by the equivariant Gromov-Witten theory of X. The equivariant theory is discussed at the end of Chapter 2.

2. Frobenius manifolds over R

Let R be a commutative algebra over $\mathbb{C}$. An (even) *Frobenius manifold defined over* R is a quadruple $(M, g, A, \mathbf{1})$ where

- M is a smooth R-scheme of relative dimension m,
- g is an R-linear, symmetric, non-degenerate quadratic form on the tangent bundle TM over R,
- A is R-linear symmetric tensor, $A : TM \otimes TM \otimes TM \to \mathcal{O}_M$,
- $\mathbf{1}$ is a vector field on M over R,

satisfying the following conditions:

(i) *Flatness*: g is a flat metric,
(ii) *Potential*: M is covered by open sets U each equipped with a commuting basis of g-flat vector fields,

$$X_1, \ldots, X_m \in \Gamma(U, TM),$$

and a potential function $\Phi \in \Gamma(U, \mathcal{O}_U)$ such that

$$A(X_i, X_j, X_k) = X_i X_j X_k(\Phi),$$

(iii) *Associativity*: the $*$-product determined by g and A is associative,
(iv) *Unit*: $\mathbf{1}$ is a g-flat unit vector field.

The complex Frobenius manifolds studied in Chapter 1 are simply (holomorphic) Frobenius manifolds over $\mathbb{C}$.

Let $\mathcal{F}_R = (M, g, A, \mathbf{1})$ be a Frobenius manifold over R. Conformal structures for $\mathcal{F}_R$ are defined exactly as before: Euler fields E on M are defined by conditions (i-iii) of Section 2.3 of Chapter 1.

Let p be an R-valued point of M. Let TM_p denote the restriction of TM to p. As M is smooth over R, TM_p is a projective R-module. The $*$-product determines an R-algebra structure $(TM_p, *)$. The point p is *semisimple over R* if there exists an algebra isomorphism:

$$(TM_p, *) \stackrel{\sim}{=} \oplus_1^m R,$$

where the direct sum algebra structure is taken on the right.

Let R be an integral domain. Let $C(R)$ denote the algebraic closure of the quotient field of R. The point p is *geometrically semisimple* if there exists an algebra isomorphism:

$$(TM_p \otimes_R C(R), *) \stackrel{\sim}{=} \oplus_1^m C(R).$$

Canonical coordinates for Frobenius manifolds over R may be defined in the étale topology. We will require canonical coordinates only in the formal case discussed below.

If S is an R-algebra, a Frobenius manifold $\mathcal{F}_S$ is obtained canonically by base change:

$$\mathcal{F}_S = (M \otimes_R S,\ g \otimes_R S,\ A \otimes_R S,\ \mathbf{1} \otimes_R S).$$

3. Formal Frobenius manifolds

Let R be a commutative algebra over $\mathbb{C}$. An (even) *formal Frobenius manifold* over R is a quadruple $(M, g, A, \mathbf{1})$,

- $M = \mathrm{Spec}(R[[K^\vee]])$ is a formal manifold over R defined by the completion at the origin of a free R-module K of rank m,
- g is a formal, R-linear, symmetric, non-degenerate quadratic form on the formal tangent bundle TM over R,
- A is a formal, R-linear, symmetric tensor,

$$A : TM \otimes TM \otimes TM \to \mathcal{O}_M,$$

- $\mathbf{1}$ is a formal vector field on M over R,

satisfying the flatness, potential, associativity, and unit conditions. The formal functions on M are:

$$\Gamma(M, \mathcal{O}_M) \stackrel{\sim}{=} R[[K^\vee]].$$

The potential condition requires the existence of a formal function Φ generating A via third partial derivatives.

Formal Euler fields E determining conformal structures on formal Frobenius manifolds are defined by conditions (i-iii) of Section 2.3 of Chapter 1.

Let $\mathcal{F}_R = (M, g, A, \mathbf{1})$ be a formal Frobenius manifold over R. The origin is the only point of M. Since $TM \cong K \otimes_R \mathcal{O}_M$, the $*$-product determines an $R[[K^\vee]]$-algebra,

$$(K \otimes_R R[[K^\vee]], *)$$

which specializes to an R-algebra $(K, *)$ at the origin. $\mathcal{F}_R$ is *semisimple at the origin over* R if there exists an algebra isomorphism:

$$(K, *) \cong \oplus_1^m R,$$

where the product algebra structure is taken on the right. As before, geometric semisimplicity is defined over the algebraic closure $C(R)$.

$\mathcal{F}_R$ is *semisimple over* $R[[K^\vee]]$ if there exists an algebra isomorphism:

$$(K \otimes_R R[[K^\vee]], *) \cong \oplus_1^m R[[K^\vee]]. \tag{31}$$

The basis of idempotent vector fields $\epsilon_1, \ldots, \epsilon_m$ (unique up to permutation) then determines m formal 1-forms $\gamma_1, \ldots, \gamma_m$ by the equations:

$$\langle \epsilon_i, \gamma_j \rangle = \delta_{ij},$$

where $\langle , \rangle$ here denotes the canonical pairing:

$$TM \times T^*M \to \mathcal{O}_M.$$

The 1-forms $\gamma_1, \ldots, \gamma_m$ are sections of the formal characteristic subvariety $C \subset T^*M$. Lemma 2 is valid (with unchanged proof) in the context of formal Frobenius manifolds defined over a $\mathbb{C}$-algebra R. Therefore, there exist formal functions (unique up to constants) $u^1, \ldots, u^m \in R[[K^\vee]]$ satisfying:

$$du^j = \gamma_j, \quad \epsilon_i = \frac{\partial}{\partial u^i}.$$

The constants may be specified by requiring $u^j \in K^\vee \cdot R[[K^\vee]]$. The functions $\{u^j\}$ are *formal canonical coordinates* on M.

4. Criteria for semisimplicity

Lemmas 17 and 18 below provide basic criteria for the semisimplicity of formal Frobenius manifolds. Both are derived from the lifting Lemma 16.

Let S be a $\mathbb{C}$-algebra and let $I \subset S$ be an ideal, and let $S_n = S/I^n$ for positive integers n. Let $(A, *)$ be an S-algebra which is a free module of rank m over S. Let A_n be the S_n-algebra,

$$A_n = A \otimes_S S_n,$$

for positive n.

LEMMA 16. *If A_n is semisimple over S_n, then A_{n+1} is semisimple over S_{n+1}. Moreover, the idempotent basis of A_n has a unique lift to an idempotent basis of A_{n+1}.*

Proof. Let $\epsilon_1, \ldots, \epsilon_m$ be the lifts to A of an idempotent basis of A_n. The projections of $\epsilon_1, \ldots, \epsilon_m$ to A_{n+1} determine a free S_{n+1}-module basis of A_{n+1}. Therefore, there exist elements $x_{ik}, y_{ijk} \in I^n$ satisfying:

$$\epsilon_i * \epsilon_i = \epsilon_i + \sum_{k=1}^{m} x_{ik}\epsilon_k \ \in A_{n+1},$$

$$\epsilon_i * \epsilon_j = \sum_{k=1}^{m} y_{ijk}\epsilon_k \ \in A_{n+1},$$

where $i \neq j$ in the second equation.

Ring axioms place restrictions on the coefficients x_{ij}, y_{ijk}. For example,

$$\epsilon_i * \epsilon_j = \epsilon_j * \epsilon_i$$

in A_{n+1} implies

$$y_{ijk} = y_{jik} \tag{32}$$

for all i, j, k. Let i, j, k be distinct indices. Then,

$$(\epsilon_i * \epsilon_j) * \epsilon_k = y_{ijk}\epsilon_k,$$

$$(\epsilon_i * \epsilon_k) * \epsilon_j = y_{ikj}\epsilon_j,$$

in A_{n+1}. By commutativity and associativity, we find:

$$y_{ijk} = 0 \tag{33}$$

for distinct indices. Similarly, for indices $i \neq j$, we find:

$$(\epsilon_i * \epsilon_j) * \epsilon_i = y_{iji}\epsilon_i,$$

$$(\epsilon_i * \epsilon_i) * \epsilon_j = \sum_{k=1}^{m} y_{ijk}\epsilon_k + x_{ij}\epsilon_j,$$

in A_{n+1}. We conclude:

$$y_{ijj} + x_{ij} = 0. \tag{34}$$

After switching i and j and using (32), we obtain

$$y_{iji} + x_{ji} = 0. \tag{35}$$

Let $\epsilon_1', \ldots, \epsilon_m'$ be an arbitrary lift to A_{n+1} of the original idempotent basis of A_n:

$$\epsilon_i' = \epsilon_i + \sum_{k=1}^{m} a_{ik}\epsilon_k \ \in A_{n+1},$$

where $a_{ik} \in I^n$. In order for $\epsilon_1', \ldots, \epsilon_m'$ to form an idempotent basis of A_{n+1}, the following equations must be satisfied in A_{n+1}:

$$\epsilon_i' * \epsilon_i' = \epsilon_i,$$

$$\epsilon_i' * \epsilon_j' = 0, \quad i \neq j.$$

These equations may be rewritten as:

$$\epsilon_i + \sum_{k=1}^{m} x_{ik}\epsilon_k + 2a_{ii}\epsilon_i = \epsilon_i + \sum_{k=1}^{m} a_{ik}\epsilon_k,$$

$$\sum_{k=1}^{m} y_{ijk}\epsilon_k + a_{ji}\epsilon_i + a_{ij}\epsilon_j = 0.$$

The first equation can be uniquely solved by:

$$a_{ii} = -x_{ii},$$

$$a_{ik} = x_{ik}, \quad i \neq j.$$

The second equation is then verified using the vanishings (33-35).

Hence, there is a unique lift of the original idempotent basis of A_n to an idempotent basis of A_{n+1}. $\square$

Let R be a $\mathbb{C}$-algebra, and let $\mathcal{F}_R$ be a formal Frobenius manifold over R (following the notation of Section 3).

Lemma 17. *If $\mathcal{F}_R$ is semisimple at the origin over R, then $\mathcal{F}_R$ is semisimple over $R[[K^\vee]]$.*

Proof. The Lemma is a direct consequence of the lifting result. Assume $\mathcal{F}_R$ is semisimple at the origin over R. Let $S = R[[K^\vee]]$ and let $I \subset S$ be the maximal ideal $K^\vee \cdot R[[K^\vee]]$. Let $(A, *) = (K \otimes_R R[[K^\vee]], *)$. Then, A_1 is a semisimple over S_1 by assumption. By Lemma 16, any idempotent basis of A_1 can be lifted compatibly to A_n for all n. Since S and A are complete with respect to the ideal I, there exists an idempotent basis of $(A, *)$. $\square$

Lemma 18. *Let R be a complete local $\mathbb{C}$-algebra with maximal ideal m_R. If $\mathcal{F}_{R/m_R}$ is semisimple at the origin over R/m_R, then $\mathcal{F}_R$ is semisimple over $R[[K^\vee]]$.*

Proof. Assume $\mathcal{F}_{R/m_R}$ is semisimple at the origin over R/m_R. By Lemma 17, $\mathcal{F}_{R/m_R}$ is semisimple over $R/m_R[[K^\vee]]$. Let $S = R[[K^\vee]]$, $I = m_R[[K^\vee]]$, and $(A, *) = (K \otimes_R R[[K^\vee]], *)$. Then, A_1 is semisimple over S_1. Hence, by Lemma 16, any idempotent basis of A_1 can be lifted compatibly to A_n for all n. Since S and A are complete with respect to the ideal I, there exists an idempotent basis of $(A, *)$. $\square$

5. Gromov-Witten theory

5.1. Novikov rings. Let X be a nonsingular projective variety. A class $\beta \in H_2(X, \mathbb{Z})$ is *effective* if

$$\beta = \pi_*[C],$$

where $\pi : C \to X$ is an algebraic map and C is a complete (possibly disconnected) curve. Let $E \subset H_2(X, \mathbb{Z})$ denote the semigroup of *effective* classes. Let $\mathbb{C}[E]$ be the semigroup ring determined by E. Since $0 \in E$, $\mathbb{C}[E]$ has a unit element. For $\beta \in E$, the corresponding element of $\mathbb{C}[E]$ will be denoted by Q^β.

Let $E^* \subset E$ denote the set of non-zero elements. Let $I \subset \mathbb{C}[E]$ denote the ideal generated by E^*. If $\beta \in E^*$, then $-\beta \notin E^*$. Hence, I is a proper maximal ideal. Two basic properties hold for classes $\beta \in E$.

Lemma 19. *Let $\beta \in E$, then*
(i) *$Q^\beta \notin I^n$ for $n >> 0$,*
(ii) *$x + y = \beta$ has finitely many solutions for $x, y \in E$.*

Proof. A projective embedding of $X \subset \mathbf{P}^r$ induces a non-negative degree function on E. As elements of E^* have positive degree, property (i) is deduced immediately. Property (ii) is obtained from the finiteness result for the degree function proven in the following Lemma. $\square$

Lemma 20. *Let $d > 0$. There are only finitely elements $\beta \in E$ of degree d.*

Proof. It suffices to prove that there are finitely many elements $\beta \in E$ of degree d represented by maps $\pi : C \to E$ where C is a nonsingular, irreducible curve and π is birational. The genus g of a birational degree d embedding in projective space satisfies:

$$g \leq (d - 1)(d - 2)/2.$$

The map π represents a point $[\pi] \in \overline{M}_g(X, \beta) \subset \overline{M}_g(\mathbf{P}^r, d)$ where β is of degree d. The disjoint union of such moduli spaces of maps to X constitutes a subscheme of the moduli space $\overline{M}_g(\mathbf{P}^r, d)$:

$$\bigcup_{\beta \in E, \quad degree(\beta)=d} \overline{M}_g(X, \beta) \quad \subset \overline{M}_g(\mathbf{P}^r, d).$$

As a subscheme has finitely many components, there are finitely many possible β for each genus g. $\square$

Define the Novikov ring $N(X)$ by:

$$N(X) = \widehat{\mathbb{C}[E]}, \tag{36}$$

where the completion is taken in the I-adic topology. Alternatively, $N(X)$ may be defined by series in Q^β:

$$N(X) = \{\sum_{\beta \in E} c_\beta Q^\beta \ | c_\beta \in \mathbb{C}\}. \tag{37}$$

Definitions (36) and (37) are proven to determine isomorphic rings by Lemmas 19-20. Multiplication is well-defined in the series ring by property (i) above of Lemma 19.

Lemma 21. *If $H_2(X, \mathbb{Z})$ is torsion free, then $\mathbb{C}[E]$ and $N(X)$ are integral domains.*

Proof. If $H_2(X, \mathbb{Z}) \stackrel{\sim}{=} \mathbb{Z}^k$, then $\mathbb{C}[H_2(X, \mathbb{Z})]$ is easily seen to be an integral domain. As $\mathbb{C}[E]$ is a subring, $\mathbb{C}[E]$ is also an integral domain. $N(X)$ is then proven be an integral domain by using the series definition (37) and the degree function on E induced by the embedding $X \subset \mathbf{P}^r$. The products of the lowest degree elements of non-zero series are non-zero. $\square$

If $X = \mathbf{P}^r$, then $E = \{n[L] \mid n \geq 0\}$ where $L \subset \mathbf{P}^r$ is a line. Then,

$$\mathbb{C}[E] = \mathbb{C}[Q^{[L]}],$$

$$N(\mathbf{P}^r) = \mathbb{C}[[Q^{[L]}]].$$

In the projective space case, we will often use the abbreviated notation $Q = Q^{[L]}$.

5.2. Canonical Frobenius structures. The genus 0 Gromov-Witten theory of X determines a formal Frobenius manifold,

$$\mathcal{F}(X) = (M, g, A, \mathbf{1}),$$

defined over the ring $N(X)$.

Let $H^*(X,\mathbb{C})$ denote the cohomology of X. We will assume the cohomology is even to avoid a discussion of superstructure. Let

$$K = H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} N(X)$$

be a free $N(X)$-module. Then,

- $M = \mathrm{Spec}(N(X)[[K^\vee]])$, the formal completion of the module K at the origin.

The dimension of M over $N(X)$ equals the rank of $H^*(X,\mathbb{C})$.

The space of formal vector fields, $\Gamma(M, TM)$, is canonically isomorphic to $K \otimes_{\mathbb{C}} \mathcal{O}_M$. As the cohomology of X is even, the intersection pairing defines a symmetric and nondegenerate (by Poincaré duality) bilinear form on $H^*(X,\mathbb{C})$:

- g is defined on TM by the $\mathcal{O}_M$-linear extension of the intersection pairing:
$$\langle \phi_1, \phi_2 \rangle = \int_X \phi_1 \cup \phi_2,$$
for $\phi_i \in K \otimes_{\mathbb{C}} \mathcal{O}_M$.

Since g has constant coefficients for the fields determined by $H^*(X,\mathbb{C})$, g is a flat metric.

The Gromov-Witten potential $F_0^X(Q,t)$ is the generating series of genus 0 Gromov-Witten invariants of X. Let $T_1, \ldots, T_m$ be a basis of $H^*(X,\mathbb{C})$ consisting of integral classes of pure dimension, and let $t^1, \ldots, t^m$ denote the corresponding formal coordinates on M. Let

$$\gamma = \sum_{i=1}^m t^i T_i.$$

In these coordinates,

$$F_0^X(Q,t) = \sum_{n\geq 0} \sum_{\beta \in E} \frac{Q^\beta}{n!} \langle \underbrace{\gamma, \ldots, \gamma}_{n} \rangle_{0,n,\beta}^X, \tag{38}$$

where the unstable degree 0 terms with $n < 3$ are omitted in the sum. The brackets denote integration over the moduli space of maps,

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{g,n,\beta}^X = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \mathrm{ev}_1^*(\gamma_1) \ldots \mathrm{ev}_n^*(\gamma_n). \tag{39}$$

Let $T_{\alpha_1}, \ldots, T_{\alpha_l}$ be the cohomology basis elements of $H^*(X,\mathbb{C})$ spanning $H^2(X,\mathbb{C})$. Let $T_{\alpha'_1}, \ldots, T_{\alpha'_{m-l}}$ denote the other elements of the cohomology basis. Let $(t, e^{\pm t^\alpha})$ denote the set:

$$(t^1, \ldots, t^m, e^{t^{\alpha_1}}, e^{-t^{\alpha_1}}, \ldots, e^{t^{\alpha_l}}, e^{-t^{\alpha_l}}).$$

Let t' denote the variables $(t^{\alpha'_1}, \ldots, t^{\alpha'_{m-l}})$.

LEMMA 22. $F_0^X(Q,t) \in N(X)[t, e^{\pm t^\alpha}]$.

Proof. Let $\beta \in E$ be a nonzero class. Then,

$$Q^\beta \sum_{n \geq 0} \frac{1}{n!} \langle \gamma, \ldots, \gamma \rangle_{0,n,\beta}^X = Q^\beta p(t') \prod_{i=1}^{l} e^{t^{\alpha_i} \int_\beta T_{\alpha_i}}, \tag{40}$$

where $p(t')$ is a polynomial. Equation (40) is a consequence of dimension constraints, the fundamental class axiom, and the divisor equation of Gromov-Witten theory. Since $\int_\beta T_{\alpha_i} \in \mathbb{Z}$, the Lemma is proven. $\square$

The basis elements T_i canonically determine a basis of formal vector fields on M. The series $F_X^0(Q,t)$ is the potential function for the formal Frobenius manifold $\mathcal{F}(X)$:

- $A(T_{i_1}, T_{i_2}, T_{i_3})$ is defined by the third partial derivatives the potential, $\partial^3 F_0^X(Q,t)/\partial t^{i_1} \partial t^{i_2} \partial t^{i_3}$.

The last structure of $\mathcal{F}(X)$ is the unit field:

- $\mathbf{1}$ is defined to be the vector field of M corresponding to the unit element of $H^*(X, \mathbb{C})$.

We will usually assume the first basis element of $H^*(X, \mathbb{C})$ is the unit element.

$\mathcal{F}(X)$ determines a formal complex Frobenius manifold. The flatness and potential conditions hold by definition. The associativity and unit conditions follow respectively from the WDVV equations and the fundamental class axiom of Gromov-Witten theory.

5.3. Flat coordinates and fundamental solutions. Let $\mathcal{F}(X)$ be the formal Frobenius manifold associated to X. Formal Levi-Civita and Dubrovin connections are defined on $\mathcal{F}(X)$ by the formulas of Section 2.1 of Chapter 1.

A fundamental solution matrix $S_{ab}(z,t)$ for ∇_z-flat formal vector fields on M may be expressed in terms of the genus 0 gravitational descendent invariants of X:

$$S_{ab} = g_{ab} + \sum_{n \geq 0,\ \beta,\ (n,\beta) \neq (0,0)} \frac{Q^\beta}{n!} \langle T_a, \frac{T_b}{z - \psi}, \gamma, \ldots, \gamma \rangle_{0,2+n,\beta}^X \tag{41}$$

where $1 \leq a, b \leq m$. We follow here the coordinate notations of Section 2.2 of Chapter 1 and Section 5.2.

A cotangent line class ψ appears in (41) at the second marking. The S matrix may be written explicitly in terms of descendent invariants:

$$S_{ab} = g_{ab} + \sum \sum_{k \geq 0} \frac{Q^\beta}{n!} z^{-k-1} \langle T_a, \tau_k(T_b), \gamma, \ldots, \gamma \rangle_{0,2+n,\beta}^X$$

with the same summation conventions on n and β. The descendent $\tau_k(T_b)$ indicates an insertion of the class $\psi^k \mathrm{ev}^*(T_b)$ in the integrand in definition (39).

A basis ∇_z-flat vector field is obtained from the raised matrix

$$S_b^c = \sum_a S_{ab} g^{ac}.$$

The flatness equations,

$$\nabla_z \sum_c S_a^c \partial_c = 0$$

are proven by the genus 0 topological recursion relations of Gromov-Witten theory. Derivations can be found, for example, in [**3**], [**6**], and [**16**]. The matrix coefficients S_{ab} are formal functions in the ring

$$N(X)[[z^{-1}, t, e^{\pm t^\alpha}]]$$

satisfying equation (4) — the proof of equation (4) is valid in the formal context.

The fundamental solution (41) will play two roles. First, the solution will be used explicitly in the study of Frobenius manifolds obtained from equivariant Gromov-Witten theory in Chapter 7. Second, the solution will motivate the definition of a J-calibration for Frobenius manifolds in Part 2.

The J-calibration obtained from S is simply the raised matrix series in $1/z$,

$$S_i^j = \sum_{n=0}^{\infty} (J_n)_i^j z^{-n}.$$

The J-calibration defines an endomorphism series in $1/z$,

$$J(z,t) = \sum_{n=0}^{\infty} J_n z^{-n}, \quad J_n(t) : TM \to TM,$$

by

$$\langle \partial_j, J_n \partial_i \rangle = \sum_k (J_n)_i^k g_{kj}.$$

The initial conditions of S imply the initial term in the endomorphism series is the *identity*,

$$J_0 = 1.$$

A direct calculation yields the *normalization* condition,

$$(J_1)_1^j = t^j.$$

By Lemma 1, the J-calibration satisfies the unitary condition:

$$J(1/z, u)J^{\dagger}(-1/z, u) = 1,$$

where the adjoint is taken with respect to the metric.

A J-calibration for a Frobenius manifold will be defined in Part 2 to be a series expansion of S_i^j in $1/z$ which satisfies the identity and normalization conditions. Frobenius manifolds obtained from Gromov-Witten theory carry canonical J-calibrations obtained the $1/z$ expansion (41).

5.4. Conformal structures. If X is a nonsingular projective variety, $\mathcal{F}(X)$ is equipped with a canonical Euler field:

$$E = \sum_{i=1}^{m} (1 - \delta(T_i))t^i \partial_i + \sum_{i=1}^{m} c_i \partial_i. \tag{42}$$

Here, the real dimension of the cohomology basis element T_i is $2\delta(T_i)$, and

$$c_1(TX) = \sum_{i=1}^{m} c_i T_i.$$

Of course, $c_i = 0$ unless $\delta(T_i) = 1$.

E is an Euler field for $\mathcal{F}(X)$ with constants:

$$D = \dim_{\mathbb{C}}(X), \quad r = 1, \quad v = -1.$$

The Lie derivatives $\mathcal{L}_E(g)$ and $\mathcal{L}_E(*)$ are determined by equation (5) and the dimension formula for the moduli space of maps,

$$\dim_{\mathbb{C}}\Big(\overline{M}_{0,n}(X, \beta)\Big) = \int_{\beta} c_1(TX) + \dim_{\mathbb{C}}(X) + n - 3.$$

The Lie derivative $\mathcal{L}_E(V)$ may be computed directly.

5.5. Bounded type. Let X be a nonsingular projective variety and let E be the semigroup of effective curve classes. For $l \geq 0$, let $E_l \subset E$ be the set:

$$E_l = \{\beta \in E \mid \dim_{\mathbb{C}}(\overline{M}_{0,3}(X, \beta)) < l\}.$$

X is of *bounded type* if E_l is a finite set for all l. Projective spaces, flag varieties, and Fano toric varieties are all of bounded type.

Let X be of bounded type. Let $\zeta \in \mathbb{C}$. Then, by dimension constraints, specialization of Q to ζ in the Gromov-Witten potential yields a power series:

$$F_0^X(Q = \zeta, t) \in \mathbb{C}[[t]]. \tag{43}$$

A formal Frobenius manifold over $\mathbb{C}$,

$$\mathcal{F}_\zeta(X) = (\mathrm{Spec}(\mathbb{C}[[H^*(X,\mathbb{C})^\vee)]], g, A(Q=\zeta), \mathbf{1}),$$

is defined for each ζ by the specialized potential (43).

5.6. Convergence. Let X be of bounded type. If the potential $F_0^X(\zeta, t)$ *converges* in a neighborhood of the origin in $H^*(X,\mathbb{C})$, then $\mathcal{F}_\zeta(X)$ determines a conformal complex Frobenius manifold.

Proposition 1. *For $\zeta \neq 0$, $\mathcal{F}_\zeta(\mathbf{P}^m)$ is a semisimple conformal complex Frobenius manifold well-defined in a neighborhood of the origin of $H^0(\mathbf{P}^m,\mathbb{C})$.*

Proof. The potential, $\mathcal{F}_0^{\mathbf{P}^m}(\zeta, t)$, is uniquely determined by the WDVV equations from the Gromov-Witten invariants of degrees 0 and 1 [**14**], [**17**]. An analytic proof of the convergence of $F_0^{\mathbf{P}^m}(\zeta, t)$ near the origin in $H^*(\mathbf{P}^m,\mathbb{C})$ using the WDVV equations is given in [**15**]. Hence, $\mathcal{F}_\zeta(X)$ defines a complex Frobenius manifold near the origin.

The canonical Euler field, E, is well-defined on the entire space $H^*(\mathbf{P}^m,\mathbb{C})$ and therefore determines a conformal structure on $\mathcal{F}_\zeta(X)$.

Let $H \in H^2(\mathbf{P}^m,\mathbb{C})$ denote the hyperplane class. Let

$$H^0, H^1, \ldots, H^m$$

define a basis of $H^*(\mathbf{P}^m,\mathbb{C})$, and let $t^0, \ldots, t^m$ denote the associated coordinates. At a point $p \in H^*(\mathbf{P}^m,\mathbb{C})$ with coordinates $p^i = 0$ for $i \neq 1$, the $*$-product on TM_p is well-known to yield the algebra:

$$\mathbb{C}[H]/(H^{m+1} - \zeta e^{p^1}). \tag{44}$$

See, for example, [**5**]. As the algebra (44) is semisimple at $p = 0$, $\mathcal{F}_\zeta(X)$ is semisimple. $\square$

6. Equivariant Gromov-Witten theory

Let $\mathbf{T}$ be an algebraic torus. Let $\mathbf{R}$ denote the equivariant cohomology ring of $\mathbf{T}$ with $\mathbb{C}$-coefficients. Let X be a nonsingular projective variety equipped with an algebraic $\mathbf{T}$-action. We will assume the $\mathbf{T}$-equivariant cohomology ring $H^*_{\mathbf{T}}(X,\mathbb{C})$ is a free $\mathbf{R}$-module — a condition which certainly holds for the standard torus actions on projective spaces, flag varieties, and toric varieties.

Let $N_{\mathbf{T}}(X) = N(X) \otimes_{\mathbb{C}} \mathbf{R}$ be the Novikov ring with $\mathbf{R}$ coefficients. Let $K_{\mathbf{T}} = H^*_{\mathbf{T}}(X,\mathbb{C}) \otimes_{\mathbf{R}} N_{\mathbf{T}}(X)$. The $\mathbf{T}$-action on X canonically determines a formal Frobenius manifold defined over $N_{\mathbf{T}}(X)$:

$$\mathcal{F}_{\mathbf{T}}(X) = (\mathrm{Spec}(N_{\mathbf{T}}(X)[[K^\vee_{\mathbf{T}}]]), g, A, \mathbf{1}), \tag{45}$$

where g is determined by the equivariant intersection pairing, A is determined by the third derivatives of the genus 0 equivariant Gromov-Witten potential, and $\mathbf{1}$ is the unit field. The flatness and potential conditions for $\mathcal{F}_{\mathbf{T}}(X)$ hold by definition. The associativity and unit conditions hold respectively by the WDVV equations and the fundamental class axiom of equivariant Gromov-Witten theory.

The construction of the Euler field for the formal Frobenius manifold $\mathcal{F}(X)$ depends upon the dimension constraint in (non-equivariant) Gromov-Witten theory. As there is no dimension constraint in equivariant Gromov-Witten theory, a corresponding Euler field can not be constructed on $\mathcal{F}_{\mathbf{T}}(X)$. The conformal structure is lost in the equivariant theory.

Localization in equivariant cohomology will play an important role in Givental's study. Let $\mathbf{R}^*$ denote the quotient field of $\mathbf{R}$. Let

$$N^*_{\mathbf{T}}(X) = N_{\mathbf{T}}(X) \otimes_{\mathbf{R}} \mathbf{R}^*.$$

Since $H^*_{\mathbf{T}}(X, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*$ is always a free $\mathbf{R}^*$-module,

$$K^*_{\mathbf{T}} = K_{\mathbf{T}} \otimes_{N_{\mathbf{T}}(X)} N^*_{\mathbf{T}}(X)$$

is always a free $N^*_{\mathbf{T}}(X)$-module. The free module assumptions needed in the equivariant theory above are not required for the localized construction.

A formal Frobenius manifold over $N^*_{\mathbf{T}}(X)$ is determined by localized equivariant data:

$$\mathcal{F}^*_{\mathbf{T}}(X) = (\mathrm{Spec}(N^*_{\mathbf{T}}[[K^*_{\mathbf{T}}(X)^\vee]]), g, A, \mathbf{1}),$$

The formal Frobenius manifold $\mathcal{F}^*_{\mathbf{T}}(X)$ is the most natural setting for the study of equivariant Gromov-Witten theory via torus localization.

CHAPTER 3

Localization

1. T-actions

Let $\mathbf{T} = \Pi_{i=0}^{m}(\mathbb{C}^*)$ be an algebraic torus. Let χ_i be the equivariant first Chern class of the dual of the standard representation of the i^{th} factor $\mathbb{C}^*$. Let χ denote the set $\{\chi_0, \ldots, \chi_m\}$. A presentation of the equivariant cohomology ring of $\mathbf{T}$ is determined by

$$\mathbf{R} = \mathbb{C}[\chi].$$

As before, let $\mathbf{R}^*$ denote the quotient field.

Let X be a nonsingular projective variety with an algebraic $\mathbf{T}$-action. Givental's study of the higher genus $\mathbf{T}$-equivariant Gromov-Witten theory on X requires two conditions:

(i) The $\mathbf{T}$-action has a finite number of 0 dimensional orbits.
(ii) The $\mathbf{T}$-action has a finite number of 1 dimensional orbits.

Maximal torus actions on algebraic homogeneous spaces and nonsingular toric varieties certainly satisfy (i-ii). If conditions (i-ii) hold, the virtual localization formula for Gromov-Witten theory yields an optimal result: the $\mathbf{T}$-equivariant Gromov-Witten invariants of X are expressed in terms of graph sums of products of integrals over Deligne-Mumford moduli spaces of stable pointed curves [**13**], [**11**].

We will describe the localization result in Gromov-Witten theory for the standard torus action on projective space, $X = \mathbf{P}^m$ (see also [**13**], [**11**]).

2. Localization

Let V be a *nonsingular* algebraic variety (or Deligne-Mumford stack) equipped with an algebraic $\mathbf{T}$-action. The localization formula expresses equivariant integrals over V as a sum of contributions over the $\mathbf{T}$-fixed subloci.

Let $H^*_{\mathbf{T}}(V, \mathbb{C})$ denote the equivariant cohomology of V. The equivariant cohomology ring $H^*_{\mathbf{T}}(V, \mathbb{C})$ is canonically an $\mathbf{R}$-module. Let

$$H^*_{\mathbf{T}}(V, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*$$

be the localization of the $\mathbf{R}$-module.

Let $\{V_i^f\}$ be the connected components of the $\mathbf{T}$-fixed locus, and let
$$\iota : \cup_i V_i^f \to V$$
denote the inclusion morphism. The nonsingularity of V implies that each V_i^f is also nonsingular [**12**]. Let N_i denote the normal bundle of V_i^f in V, and let $e(N_i)$ denote the equivariant Euler class (top Chern class) of N_i.

The localization formula [**1**] is:
$$[V] = \iota_* \sum_i \frac{[V_i^f]}{e(N_i)} \quad \in H^*_{\mathbf{T}}(V, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*. \tag{46}$$
The formula is well-defined as the Euler classes $e(N_i)$ are invertible in localized equivariant cohomology.

Let $\xi \in H^*_{\mathbf{T}}(V, \mathbb{C})$ be a class of degree equal to (twice) the dimension of V. The Bott residue formula [**2**] expresses integrals over V in terms of fixed point data:
$$\int_V \xi = \sum_i \int_{V_i^f} \frac{\iota^*(\xi)}{e(N_i)}.$$
The Bott residue formula is an immediate consequence of (46). Localization therefore provides an effective method of computing integrals over V when the fixed loci V_i^f are well-understood.

3. T-actions on projective spaces

Let $\mathbf{T}$ act on the vector space $W = \oplus_{i=0}^m \mathbb{C}$ by the diagonal representation. A $\mathbf{T}$-action on
$$\mathbf{P}(W) = \mathbf{P}^m$$
is canonically obtained. The $\mathbf{T}$-action lifts canonically to $\mathcal{O}_{\mathbf{P}^m}(1)$. Let $H \in H^*_{\mathbf{T}}(\mathbf{P}^m, \mathbb{C})$ be the equivariant first Chern class of $\mathcal{O}_{\mathbf{P}^m}(1)$. The standard presentation of $H^*_{\mathbf{T}}(\mathbf{P}^m, \mathbb{C})$ is:
$$H^*_{\mathbf{T}}(\mathbf{P}^m, \mathbb{C}) \stackrel{\sim}{=} \mathbb{C}[H, \chi]/(\Pi_{i=0}^m (H - \chi_i)). \tag{47}$$

The fixed points $\{p_0, \ldots, p_m\}$ of the $\mathbf{T}$-action on $\mathbf{P}^m$ correspond to the canonical basis vectors in W. The 1-dimensional orbits of $\mathbf{T}$ are the lines L_{ij} connecting p_i and p_j.

Define localized cohomology classes $\phi_i \in H^*_{\mathbf{T}}(\mathbf{P}^m, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*$ by:
$$\phi_i = \frac{[p_i]}{e(T_{p_i})},$$
where $[p_i]$ is (the dual of) the equivariant fundamental class of the point p_i and T_{p_i} is the rank m equivariant tangent space of p_i. The classes

$\phi_0, \ldots, \phi_m$ determine a basis of the the localized ring $H^*_{\mathbf{T}}(\mathbf{P}^m, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*$ over $\mathbf{R}^*$.

4. T-actions on $\overline{M}_{g,n}(\mathbf{P}^m, d)$

There is a canonically induced $\mathbf{T}$-action on the stack of stable maps $\overline{M}_{g,n}(\mathbf{P}^m, d)$. The torus acts on a stable map to $\mathbf{P}^m$ by translating the image. Following [**13**], we can identify the components of the $\mathbf{T}$-fixed locus of $\overline{M}_{g,n}(\mathbf{P}^m, d)$ with a set of graphs.

A graph $\Gamma \in G_{g,n}(\mathbf{P}^m, d)$ consists of the data $(V, E, N, \gamma, j, \delta)$ where:

(i) V is the vertex set,
(ii) $\gamma : V \to \mathbb{Z}_{\geq 0}$ is a genus assignment,
(iii) $j : V \to \{0, \ldots, m\}$ is a function,
(iv) E is the edge set,
  (a) If an edge e connects $v, v' \in V$, then $j(v) \neq j(v')$, in particular, there are no self edges,
  (b) Γ is connected,
(v) $\delta : E \to \mathbb{Z}_{>0}$ is a degree assignment,
(vi) $N = \{1, \ldots, n\}$ is a set of markings incident to vertices,
(vii) $g = \sum_{v \in V} \gamma(v) + h^1(\Gamma)$,
(viii) $d = \sum_{e \in E} \delta(e)$.

The components of the $\mathbf{T}$-fixed point set of $\overline{M}_{g,n}(\mathbf{P}^m, d)$ are in bijective correspondence with the graph set $G_{g,n}(\mathbf{P}^m, d)$. The correspondence is valid for $d = 0$, but here the graphs consist of single edgeless vertices.

Let $\pi : (C, p_1, \ldots, p_n) \to \mathbf{P}^m$ be a $\mathbb{C}^*$-fixed stable map. The images of all marked points, nodes, contracted components, and ramification points must lie in the $\mathbf{T}$-fixed point set $\{p_0, \ldots, p_m\}$ of $\mathbf{P}^m$. Each non-contracted irreducible component $D \subset C$ must lie over a fixed line L_{ij}. D may be ramified only over the two fixed points $\{p_i, p_j\}$. Therefore D must be nonsingular and rational. Moreover, the restriction $\pi|_D$ is uniquely determined by the degree $deg(\pi|_D)$, $\pi|_D$ must be the rational Galois cover with full ramification over p_i and p_j.

To an invariant stable map $\pi : (C, p_1, \ldots, p_n) \to \mathbf{P}^m$, we associate a graph $\Gamma \in G_{g,n}(\mathbf{P}^m, d)$ as follows:

(i) V is the set of connected components of $\pi^{-1}(\{p_0, \ldots, p_m\})$,
(ii) $\gamma(v)$ is the arithmetic genus of the component corresponding to v (taken to be 0 if the component is an isolated point),
(iii) $j(v)$ is defined by $\pi(v) = p_{j(v)}$,
(iv) E is the set of non-contracted irreducible components $D \subset C$,
(v) $\delta(D) = deg(\pi|_D)$,
(vi) N is the marking set.

Conditions (vii-viii) hold by definition.

The set of $\mathbf{T}$-fixed stable maps with a given graph Γ is naturally identified with a finite quotient of a product of moduli spaces of pointed curves. Define:

$$\overline{M}_\Gamma = \prod_{v \in V} \overline{M}_{\gamma(v), val(v)}.$$

The valence $val(v)$ is the number of incident edges and markings. $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ are interpreted as points in this product. Over $\overline{M}_\Gamma$, there is a canonical universal family of $\mathbb{C}^*$-fixed stable maps,

$$\rho : U \to \overline{M}_\Gamma,$$

$$\pi : U \to \mathbf{P}^m,$$

yielding a morphism of stacks $\tau_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}(\mathbf{P}^m, d)$.

There is a natural automorphism group $\mathbf{A}$ acting equivariantly on U and $\overline{M}_\Gamma$ with respect to the morphisms ρ and π. $\mathbf{A}$ acts via automorphisms of the Galois covers (corresponding to the edges) and the symmetries of the graph Γ. $\mathbf{A}$ is filtered by an exact sequence of groups,

$$1 \to \prod_{e \in E} \mathbb{Z}/\delta(e) \to \mathbf{A} \to \mathrm{Aut}(\Gamma) \to 1,$$

where $\mathrm{Aut}(\Gamma)$ is the automorphism group of Γ: $\mathrm{Aut}(\Gamma)$ is the subgroup of the permutation group of the vertices and edges which respects all the structures of Γ. $\mathrm{Aut}(\Gamma)$ acts naturally on $\prod_{e \in E} \mathbb{Z}/\delta(e)$ and $\mathbf{A}$ is the semidirect product.

Let Q_Γ denote the quotient stack $\overline{M}_\Gamma/\mathbf{A}$. $\mathbb{Q}_\Gamma$ is a nonsingular Deligne-Mumford stack. The induced map:

$$\tau_\Gamma/\mathbf{A} : Q_\Gamma \to \overline{M}_{g,n}(\mathbf{P}^m, d)$$

is a closed immersion of Deligne-Mumford stacks.

The moduli space $\overline{M}_{g,n}(\mathbf{P}^m, d)$ may be singular and non-reduced in the case $g > 0$. Therefore, the $\mathbf{T}$-fixed substack is not guaranteed to be nonsingular and reduced. However, via an analysis of the equivariant perfect obstruction theory of $\overline{M}_{g,n}(\mathbf{P}^m, d)$, the substack Q_Γ is proven in [**11**] to be a component of the $\mathbf{T}$-*fixed substack* of $\overline{M}_{g,n}(\mathbf{P}^m, d)$.

PROPOSITION 2. *The connected components of the* $\mathbf{T}$*-fixed substack of the moduli space* $\overline{M}_{g,n}(\mathbf{P}^m, d)$ *are in bijective correspondence with the graph set* $G_{g,n}(\mathbf{P}^m, d)$ *by the association* $Q_\Gamma \leftrightarrow \Gamma$.

Components of the $\mathbf{T}$-fixed substack of $\overline{M}_{g,n}(X, \beta)$ may also be described by graph data in case the $\mathbf{T}$-action on X satisfies conditions (i-ii) of Section 1.

5. Tautological classes

Tautological classes on moduli spaces of stable curves and stable maps are required for the localization formula in Gromov-Witten theory.

Let $\mathbb{L}_i$ denote the i^{th} cotangent line bundle on the moduli space $\overline{M}_{g,n}$. The fiber of $\mathbb{L}_i$ over the moduli point $[C, p_1, \ldots, p_n] \in \overline{M}_{g,n}$ is $T^*_{C,p}$. Let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n}, \mathbb{C}).$$

Let $\mathbb{E}$ denote the Hodge bundle on $\overline{M}_{g,n}$. The fiber of $\mathbb{E}$ over the moduli point $[C, p_1, \ldots, p_n]$ is $H^*(C, \omega_C)$. Let

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{M}_{g,n}, \mathbb{C}).$$

Since the vector bundles $\mathbb{L}_i$ and $\mathbb{E}$ are well-defined on $\overline{M}_{g,n}(X, \beta)$, tautological ψ and λ classes are determined in $H^*(\overline{M}_{g,n}(X, \beta), \mathbb{C})$.

The λ classes are elementary symmetric functions of the Chern roots $\rho_1, \ldots, \rho_g$ of $\mathbb{E}$ on $\overline{M}_{g,n}(X, \beta)$. We will often write the λ classes in terms of the Chern roots.

6. The Localization formula

6.1. Virtual localization. Integrals in Gromov-Witten theory are always taken against the virtual class $[\overline{M}_{g,n}(X, \beta)]^{vir}$ of the moduli space of maps $\overline{M}_{g,n}(X, \beta)$. As the moduli space $\overline{M}_{g,n}(X, \beta)$ may be singular and non-reduced, the localization formula does not directly apply.

However, the perfect obstruction theory of $\overline{M}_{g,n}(X, \beta)$ together with the virtual class may be viewed as defining a virtual smooth structure on the moduli space of maps. A localization formula for the equivariant virtual class of $\overline{M}_{g,n}(X, \beta)$ is proven in [**11**]. For the $\mathbf{T}$-action on $\mathbf{P}^m$,

$$[\overline{M}_{g,n}(\mathbf{P}^m, d)]^{vir} = \sum_{\Gamma \in G_{g,n}(\mathbf{P}^m, d)} \frac{1}{|A_\Gamma|} \frac{\tau_{\Gamma *}[\overline{M}_\Gamma]}{e(N_\Gamma^{vir})} \tag{48}$$

in localized equivariant Chow theory, $A_*^{\mathbf{T}}(\overline{M}_{g,n}(\mathbf{P}^m, d) \otimes_{\mathbf{R}} \mathbf{R}^*$. The $\mathbf{T}$-fixed loci Q_Γ enter (48) as push-forwards of $\overline{M}_\Gamma$ via τ_Γ.

The Euler class of the normal complex, $e(N_\Gamma^{vir})$, is specified by the equivariant perfect obstruction theory of $\overline{M}_{g,n}(\mathbf{P}^m, d)$. A complete derivation may be found in [**11**]. Our goal here is to explain the answer in terms of the data of the graph Γ.

Let $\Gamma \in G_{g,n}(\mathbf{P}^m, d)$. We will identify the τ_Γ pull-back of $1/e(N_\Gamma^{vir})$ to $\overline{M}_\Gamma$,

$$\overline{M}_\Gamma = \prod_{v \in V} \overline{M}_{\gamma(v),val(v)}. \tag{49}$$

We will find:

$$\tau_\Gamma^*(\frac{1}{e(N_\Gamma^{vir})}) = \prod_{v\in V} \frac{1}{\tilde{N}_\Gamma(v)} \cdot \prod_{e \in E} \frac{1}{\tilde{N}_\Gamma(e)}, \tag{50}$$

where the *vertex* and *edge contributions*, $1/\tilde{N}_\Gamma(v)$ and $1/\tilde{N}_\Gamma(e)$, lie in localized equivariant cohomology rings:

$$\frac{1}{\tilde{N}_\Gamma(v)} \in H^*_{\mathbf{T}}(\overline{M}_{\gamma(v),val(v)}, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*,$$

$$\frac{1}{\tilde{N}_\Gamma(e)} \in \mathbf{R}^*.$$

6.2. Vertex contributions. There are four types of vertices which we will treat independently here. In integration formulas, a uniform treatment of the four types is often found.

A vertex v is *stable* if $2\gamma(v) - 2 + val(v) > 0$. If v is stable, the moduli space $\overline{M}_{\gamma(v),val(v)}$ is a factor of $\overline{M}_\Gamma$ by (49). The intermediate contribution $1/\tilde{N}_\Gamma(v)$ will be an equivariant cohomology class on the factor $\overline{M}_{\gamma(v),val(v)}$ in this case.

• Let v be a stable vertex. Let $e_1, \ldots, e_l$ denote the distinct edges incident to v (in bijective correspondence to a subset of the (local) markings of the moduli space $\overline{M}_{\gamma(v),val(v)}$). Let e_i connect v to the vertex v_i. Let ψ_i denote the cotangent line of the marking at v corresponding to e_i.

$$\begin{aligned} \frac{1}{\tilde{N}_\Gamma(v)} \quad &= \quad \frac{1}{e(T_{p_{j(v)}})} \cdot \prod_{i=1}^{l} \frac{1}{\frac{\chi_{j(v)} - \chi_{j(v_i)}}{\delta(e_i)} - \psi_i} \cdot \\ & \quad \prod_{k=1}^{\gamma(v)} \prod_{j \neq j(v)} \left((\chi_{j(v)} - \chi_j) - \rho_k \right). \end{aligned}$$

Both the tautological ψ and λ classes enter in $1/\tilde{N}_\Gamma(v)$. The Gromov-Witten theory of $\mathbf{P}^m$ is therefore fundamentally related to the intersection theory of the moduli space of curves.

If v is an unstable vertex, then $\gamma(v) = 0$ and $val(v) \leq 2$. There are three unstable cases: two with valence 2 and one with valence 1.

• Let v be an *unmarked* vertex with $\gamma(v)=0$ and $val(v)=2$. Let e_1 and e_2 be the two incident edges connecting v to the vertices v_1 and v_2 respectively. Then:

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})} \cdot \frac{1}{\frac{\chi_{j(v)}-\chi_{j(v_1)}}{\delta(e_1)} + \frac{\chi_{j(v)}-\chi_{j(v_2)}}{\delta(e_2)}}$$

• Let v be a 1-marked vertex with $\gamma(v)=0$ and $val(v)=2$. Then:

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})}$$

there are no contributing factors.
• Let v be an *unmarked* vertex with $\gamma(v)=0$ and $val(v)=1$. Let e be the unique incident edge connecting v to the vertex v'. Then:

$$\frac{1}{\tilde{N}_\Gamma(v)} = \frac{1}{e(T_{p_{j(v)}})} \cdot \frac{\chi_{j(v)}-\chi_{j(v')}}{\delta(e)}.$$

6.3. Edge contributions. Let $e \in E$ be an edge corresponding to the non-contracted irreducible component $D \subset C$ (where

$$[\pi : (C, p_1, \ldots, p_n) \to \mathbf{P}^m]$$

is a moduli point parameterized by $\overline{M}_\Gamma$). Let v and v' be the vertices connected by e. The edge contribution is:

$$\frac{1}{\tilde{N}_\Gamma(e)} \quad = \quad \frac{e(T_{p_{j(v)}})\ e(T_{p_{j(v')}})}{(-1)^{\delta(e)}\frac{\delta(e)!^2}{\delta(e)^{2\delta(e)}}(\chi_{j(v)}-\chi_{j(v')})^{2\delta(e)}} \cdot$$

$$\prod_{j\notin\{j(v),j(v')\}} \frac{1}{\prod_{i=0}^{\delta(e)} \frac{(\delta(e)-i)(\chi_{j(v)}-\chi_j)+i(\chi_{j(v')}-\chi_j)}{\delta(e)}}.$$

6.4. Integration. Localization yields an integration formula for the $\mathbf{T}$-equivariant Gromov-Witten theory of $\mathbf{P}^m$. Let ξ be an equivariant class

$$\xi \in H^*_{\mathbf{T}}(\overline{M}_{g,n}(\mathbf{P}^m, d), \mathbb{C}).$$

The equivariant integral

$$\int_{[\overline{M}_{g,n}(\mathbf{P}^m,d)]^{vir}} \xi \tag{51}$$

is defined by

$$\int_{[\overline{M}_{g,n}(\mathbf{P}^m,d)]^{vir}} \xi = \int_{\overline{M}_{g,n}(\mathbf{P}^m,d)} [\overline{M}_{g,n}(\mathbf{P}^m, d)]^{vir} \cap \xi \quad \in \mathbf{R},$$

where the right side is the equivariant push-forward to a point. The localization formula for the virtual class directly yields a residue formula for the equivariant integral (51):

$$\int_{[\overline{M}_{g,n}(\mathbf{P}^m,d)]^{vir}} \xi = \sum_{\Gamma \in G_{g,n}(\mathbf{P}^m,d)} \frac{1}{|\mathbf{A}|} \int_{\overline{M}_\Gamma} \frac{\tau_\Gamma^*(\xi)}{\prod_{v\in V} \tilde{N}_\Gamma(v) \cdot \prod_{e \in E} \tilde{N}_\Gamma(e)}$$

The right side of the above equation only involves integration over the Deligne-Mumford moduli spaces of stable curves.

7. Gravitational descendents

The localization formula may be used to determine the the (localized) $\mathbf{T}$-equivariant descendent invariants of $\mathbf{P}^m$:

$$\langle \prod_{i=1}^n \tau_{a_i}(\phi_{b_i}) \rangle_{g,d}^{\mathbf{P}^m} = \int_{[\overline{M}_{g,n}(\mathbf{P}^m,d)]^{vir}} \prod_{i=1}^n \psi_i^{a_i} \mathrm{ev}_i^*(\phi_{b_i}). \tag{52}$$

Let ξ denote the equivariant integrand on the right side of the above equation.

The virtual residue formula determines the equivariant integral (52) in terms of tautological integrals over the moduli spaces of curves:

$$\langle \prod_{i=1}^n \tau_{a_i}(\phi_{b_i}) \rangle_{g,d}^{\mathbf{P}^m} = \sum_{\Gamma \in G_{g,n}(\mathbf{P}^m,d)} \frac{1}{|\mathbf{A}|} \int_{\overline{M}_\Gamma} \frac{\tau_\Gamma^*(\xi)}{\prod_{v\in V} \tilde{N}_\Gamma(v) \cdot \prod_{e \in E} \tilde{N}_\Gamma(e)}.$$

The pull-back of ξ to $\overline{M}_\Gamma$ factorizes over the vertices of Γ:

$$\tau_\Gamma^*(\xi) = \prod_{v \in V} \xi(v).$$

There are four types of vertex contributions $\xi(v)$.

• Let v be a stable vertex. Let $S \subset \{1, \ldots, n\}$ denote the set of markings incident to the vertex v. If $j(v) = b_i$ for all i, then

$$\xi(v) = \prod_{i \in S} \psi_i^{a_i} \quad \in H_{\mathbf{T}}^*(\overline{M}_{\gamma(v),val(v)}, \mathbb{C}).$$

If $j(v) \neq b_i$, for some i then $\xi(v) = 0$.
• Let v be an unmarked vertex with $\gamma(v) = 0$ and $val(v) = 2$. Then,

$$\xi(v) = 1.$$

• Let v be a 1-marked vertex with $\gamma(v) = 0$ and $val(v) = 2$. Let i be the unique marking incident to v. Let e denote the unique edge incident to v. Let e connect v to the vertex v'. If $j(v) = b_i$, then

$$\xi(v) = \Big(- \frac{\chi_{j(v)} - \chi_{j(v')}}{\delta(e)} \Big)^{a_i} \cdot \delta_{b_i, j(v)}.$$

If $j(v) \neq b_i$, then $\xi(v) = 0$.

• Let v be an unmarked vertex with $\gamma(v) = 0$ and $val(v) = 1$. Then,

$$\xi(v) = 1.$$

We find an explicit formula for the gravitational descendent invariants of $\mathbf{P}^m$ in terms of tautological integrals over the moduli space of curves.

Proposition 3. *The equivariant gravitational descendents of $\mathbf{P}^m$ are determined by graph sums of Hodge integrals:*

$$\langle \prod_{i=1}^{n} \tau_{a_i}(\phi_{b_i}) \rangle_{g,d}^{\mathbf{P}^m} = \sum_{\Gamma \in G_{g,n}(\mathbf{P}^m, d)} \frac{1}{|\mathbf{A}|} \int_{\overline{M}_\Gamma} \prod_{v \in V} \frac{\xi(v)}{\tilde{N}_\Gamma(v)} \cdot \prod_{e \in E} \frac{1}{\tilde{N}_\Gamma(e)}.$$

CHAPTER 4

Materialization

1. Overview

Let X be a nonsingular projective variety with an algebraic $\mathbf{T}$-action. We will assume the $\mathbf{T}$-action has finitely many 0 and 1 dimensional orbits (conditions (i-ii) of Chapter 3). The $\mathbf{T}$-action determines a decomposition of X into affine cells by the Bialynicki-Birula Theorem. The groups $H^*(X,\mathbb{Z})$ and $A^*(X,\mathbb{Z})$ are isomorphic and freely generated by the classes of the cell closures. The localized equivariant quantum cohomology yields a formal Frobenius manifold $\mathcal{F}^*_{\mathbf{T}}(X)$ defined over the ring $N^*_{\mathbf{T}}(X)$ — see Chapter 2. Here, we will prove $\mathcal{F}^*_{\mathbf{T}}(X)$ is semisimple by an explicit construction of formal canonical coordinates defined via graph sums arising in the localization formula. Givental views these graph sums as a *materialization* of canonical coordinates in equivariant quantum cohomology.

2. Semisimplicity of equivariant cohomology

Let $p_0,\ldots,p_m$ be the $\mathbf{T}$-fixed points of X. Let $e_i = e(T_{p_i})$ denote the equivariant Euler class of the tangent space representation at p_i. Let $\phi_i \in H^*_{\mathbf{T}}(X,\mathbb{C})\otimes_{\mathbf{R}}\mathbf{R}^*$ be defined by:

$$\phi_i = \frac{[p_i]}{e_i}.$$

The classes $\phi_0,\ldots,\phi_m$ determine a natural $\mathbf{R}^*$-basis of the localized equivariant cohomology ring $H^*_{\mathbf{T}}(X,\mathbb{C})\otimes_{\mathbf{R}}\mathbf{R}^*$. The equivariant intersection form in the ϕ basis is:

$$\langle \phi_i,\phi_j\rangle = \frac{\delta_{ij}}{e_i},$$

Flat coordinates on $H^*_{\mathbf{T}}(X,\mathbb{C})\otimes_{\mathbf{R}}\mathbf{R}^*$ are defined by

$$\gamma = \sum t^\mu \phi_\mu.$$

Following the conventions of Section 4.1 of Chapter 1, Greek indices will be used for the flat coordinates $\{t^\mu\}$.

The localized equivariant cohomology ring $H^*_{\mathbf{T}}(X, \mathbb{C}) \otimes_{\mathbf{R}} \mathbf{R}^*$ is semisimple over $\mathbf{R}^*$ with idempotents ϕ_i:

$$\phi_i \cdot \phi_j = \delta_{ij} \phi_i.$$

The identity element of the equivariant ring is $\sum_{i=0}^{m} \phi_i$. For notational convenience, we will denote the identity by $\phi_{\mathbf{1}}$.

The classical semisimplicity implies the semisimplicity of the Frobenius structure obtained from the localized equivariant quantum cohomology of X.

LEMMA 23. *The formal Frobenius manifold $\mathcal{F}^*_{\mathbf{T}}(X)$ is semisimple over $N^*_{\mathbf{T}}(X)[[K^*_{\mathbf{T}}(X)^\vee]]$*

Proof. $N^*_{\mathbf{T}}(X)$ is a complete local $\mathbb{C}$-algebra with maximal ideal generated by $\{Q^\beta \mid 0 \neq \beta \in E\}$ and quotient field $\mathbf{R}^*$. The formal Frobenius manifold,

$$\mathcal{F}^*_{\mathbf{T}}(X) \otimes_{N^*_{\mathbf{T}}(X)} \mathbf{R}^*,$$

obtained by tensoring with the quotient field, is semisimple over $\mathbf{R}^*$ at the origin — since the corresponding tangent algebra is the localized equivariant cohomology ring of X. The Lemma then follows from Lemma 18. $\square$

By Lemmas 16 and 18, there exists a unique idempotent basis,

$$\epsilon_0, \ldots, \epsilon_m,$$

of formal vector fields on $\mathcal{F}^*_{\mathbf{T}}(X)$ which specializes to the idempotent basis $\phi_0, \ldots, \phi_m$. Let $u^0, \ldots, u^m \in N^*_{\mathbf{T}}[[K^*_{\mathbf{T}}(X)^\vee]]$ be the corresponding formal canonical coordinates vanishing at the origin (see Section 3 of Chapter 1). We will consider the formal functions $u^i(Q, t)$ as series in the variables Q and $t^0, \ldots, t^m$ with coefficients in $\mathbf{R}^*$.

Roman indices will be reserved for the canonical coordinates $\{u^i\}$. In fact, both the flat and the canonical coordinate sets are naturally indexed by the fixed points $p_0, \ldots, p_m$.

3. Graph contributions

The materialization of canonical coordinates is obtained from graph sums in the genus 0 localization formula. For the $\mathbf{T}$-action on projective space, the localization graph sums are described explicitly in Section 6 of Chapter 3. Here, we assume X is a nonsingular projective variety equipped with a $\mathbf{T}$-action satisfying conditions (i-ii) of Chapter 3.

Let $F_0^X(Q,t)$ be the $\mathbf{T}$-equivariant genus 0 Gromov-Witten potential of X:

$$F_0^X(Q,t) = \sum_{n\geq 0}\sum_{\beta\in E}\frac{Q^\beta}{n!}\langle\underbrace{\gamma,\ldots,\gamma}_{n}\rangle_{0,n,\beta}^X.$$

For each graph $\Gamma \in G_{0,n}(X,\beta)$, let

$$\mathrm{Cont}_\Gamma(F_0^X) \in N_{\mathbf{T}}^*(X)[[t]]$$

denote the contribution of Γ to F_0^X in the localization formula,

$$F_0^X(Q,t) = \sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G_{0,n}(X,\beta)}\mathrm{Cont}_\Gamma(F_0^X).$$

4. Derivative notation

Derivatives of the potential F_0^X also determine generating series of equivariant Gromov-Witten invariants:

$$\frac{\partial^r F_0^X}{\partial t^{\mu_1}\ldots\partial t^{\mu_r}}(Q,t) = \sum_{n\geq 0}\sum_{\beta}\frac{Q^\beta}{n!}\langle\phi_{\mu_1},\ldots,\phi_{\mu_r},\gamma,\ldots,\gamma\rangle_{0,r+n,\beta}^X. \tag{53}$$

Derivatives will often be expressed by double brackets:

$$\frac{\partial^r F_0^X}{\partial t^{\mu_1}\ldots\partial t^{\mu_r}}(Q,t) = \langle\langle\phi_{\mu_1},\ldots,\phi_{\mu_r}\rangle\rangle_0^X,$$

where

$$\begin{aligned}\langle\langle\phi_{\mu_1},\ldots,\phi_{\mu_r}\rangle\rangle_0^X &= \sum_\beta Q^\beta\langle\langle\phi_{\mu_1},\ldots,\phi_{\mu_r}\rangle\rangle_{0,\beta}^X,\\ \langle\langle\phi_{\mu_1},\ldots,\phi_{\mu_r}\rangle\rangle_{0,\beta}^X &= \sum_{n\geq 0}\frac{1}{n!}\langle\phi_{\mu_1},\ldots,\phi_{\mu_r},\gamma,\ldots,\gamma\rangle_{0,r+n,\beta}^X.\end{aligned}$$

Unstable values are always omitted in the above sums.

For each graph $G_{0,r+n}(X,\beta)$, the contribution

$$\mathrm{Cont}_\Gamma(\frac{\partial^r F_0^X}{\partial t^{\mu_1}\ldots\partial t^{\mu_r}}),$$

is well-defined. Graphs contributing to (53) have r distinguished marking corresponding to the derivative insertions.

The differential operator $\partial/\partial t^{\mathbf{1}}$, defined by

$$\frac{\partial}{\partial t^{\mathbf{1}}} = \sum_{\mu=0}^m \frac{\partial}{\partial t^\mu},$$

corresponds to the insertion of the identity element $\phi_{\mathbf{1}}$.

5. Materialization

We will define four sets of formal functions via localization graph contributions.

- Let $G^{\overline{u}^i}_{0,2+n}(X,\beta) \subset G_{0,2+n}(X,\beta)$ denote the set of graphs for which markings 1 and 2 lie on a single irreducible component contracted to p_i. Let

$$\overline{u}^i(Q,t) = \sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G^{\overline{u}^i}_{0,2+n}(X,\beta)} \mathrm{Cont}_\Gamma(e_i\frac{\partial^2 F_0^X}{\partial t^i \partial t^i}).$$

  A direct calculation shows that the degree 0 term of $\overline{u}^i$ equals t^i.

- Let $G^{\partial\overline{u}^i/\partial t^\mu}_{0,3+n}(X,\beta) \subset G_{0,3+n}(X,\beta)$ denote the set of graphs for which markings 1 and 2 lie on a single irreducible component contracted to p_i. Let

$$\frac{\partial\overline{u}^i}{\partial t^\mu}(Q,t) = \sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G^{\partial\overline{u}^i/\partial t^\mu}_{0,3+n}(X,\beta)} \mathrm{Cont}_\Gamma(e_i\frac{\partial^3 F_0^X}{\partial t^i \partial t^i\partial t^\mu}).$$

  The function $\partial\overline{u}^i/\partial t^\mu$ defined above is easily seen to equal the derivative of $\overline{u}^i(Q,t)$ by t^μ — justifying the notation.

- Let $G^{\sqrt{\Delta^i}}_{0,3+n}(X,\beta) \subset G_{0,3+n}(X,\beta)$ denote the set of graphs for which markings 1, 2, and 3 lie on a single irreducible component contracted to p_i. Then,

$$\sqrt{\Delta^i}(Q,t) = \sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G^{\sqrt{\Delta^i}}_{0,3+n}(X,\beta)} \sqrt{e_i}\mathrm{Cont}_\Gamma(e_i\frac{\partial^3 F_0^X}{\partial t^i \partial t^i\partial t^i}).$$

- Let $G^{\Psi^i_\mu}_{0,3+n}(X,\beta) \subset G_{0,3+n}(X,\beta)$ denote the set of graphs for which marking 1 lies on an irreducible component E contracted to p_i *and* markings 2 and 3 lie on *distinct* branches off of E. Then,

$$\Psi^i_\mu(Q,t) = \sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G^{\Psi^i_\mu}_{0,3+n}(X,\beta)} \frac{1}{\sqrt{e_i}}\mathrm{Cont}_\Gamma(e_i\frac{\partial^3 F_0^X}{\partial t^i \partial t^\mu\partial t^{\mathbf{1}}}).$$

The main materialization results, established in Sections 6 and 7 below, provide an explicit construction of formal canonical coordinates for $\mathcal{F}^*_{\mathbf{T}}(X)$.

Proposition 4. *Three materialization results hold:*

(i) *The functions* $\overline{u}^0(Q,t), \ldots \overline{u}^m(Q,t)$ *determine formal canonical coordinates at the origin of the Frobenius manifold* $\mathcal{F}^*_{\mathbf{T}}(X)$,

$$\overline{u}^i(Q,t) = u^i(Q,t), \quad \frac{\partial}{\partial \overline{u}^i} = \epsilon_i.$$

(ii) *The functions* $\sqrt{\Delta^i}$ *are square roots of the norms of the vector fields* ϵ_i:

$$\langle \epsilon_i, \epsilon_i \rangle^{-1} = \Delta^i,$$

where $\Delta^i = (\sqrt{\Delta^i})^2$.

(iii) Ψ^i_μ *is the transition matrix between the bases* ∂_μ *and* $\tilde{\epsilon}_i$ *of vector fields on* $\mathcal{F}^*_{\mathbf{T}}(X)$, *where* $\tilde{\epsilon}_i = \sqrt{\Delta^i}\epsilon_i$.

The notation here exactly matches the conventions of Section 4.1 of Chapter 1. The metric g is determined by:

$$g_{\mu\nu} = \frac{\delta_{\mu\nu}}{e_\mu}.$$

The only difference is that the index set here for the bases is naturally $\{0, 1, \ldots, m\}$ instead of $\{1, \ldots, m\}$.

6. Local splitting equations

6.1. Integral series. Let ν_r denote the forgetful map to the moduli space of curves defined by stabilization:

$$\nu_r : \overline{M}_{0,r+n}(X,\beta) \to \overline{M}_{0,r},$$

for $r \geq 3$. Consider the following two $\mathbf{T}$-equivariant Gromov-Witten integral series,

$$\theta^\beta_{\alpha_1\alpha_2\alpha_3\alpha_4}(Q,t) = \sum_{n\geq 0} \frac{Q^\beta}{n!} \int_{[\overline{M}_{0,4+n}(X,\beta)]^{\mathrm{vir}}} \Big(\nu_4^*([D_4]) \prod_{i=1}^{4} \mathrm{ev}_i^*(\phi_{\alpha_i}) \prod_{i=1}^{n} \mathrm{ev}_{4+i}^*(\gamma)\Big), \tag{54}$$

$$\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}(Q,t) = \sum_{n\geq 0} \frac{Q^\beta}{n!} \int_{[\overline{M}_{0,5+n}(X,\beta)]^{\mathrm{vir}}} \Big(\nu_5^*([D_5]) \prod_{i=1}^{5} \mathrm{ev}_i^*(\phi_{\alpha_i}) \prod_{i=1}^{n} \mathrm{ev}_{5+i}^*(\gamma)\Big). \tag{55}$$

The following notation is used in the above definitions (54-55):

- the indices $\alpha_1, \ldots, \alpha_5$ are fixed (and allowed to equal $\mathbf{1}$),
- the moduli points $[D_4] \in \overline{M}_{0,4}$, $[D_5] \in \overline{M}_{0,5}$ are fixed,

- the classes $\nu_4^*([D_4])$, $\nu_5^*([D_5])$ are defined in $\mathbf{T}$-equivariant cohomology (by Poincaré duality).

The series $\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}(Q,t)$ and $\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{11}}(Q,t)$ can be computed by specializing the moduli points $[D_4]$ and $[D_5]$.

Lemma 24. *The following evaluations holds:*

$$\begin{aligned}\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}(Q,t) &= Q^\beta \frac{1}{e_{\alpha_1}}\delta_{0\beta}\delta_{\alpha_1\alpha_2},\\ \vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{11}}(Q,t) &= Q^\beta \langle\langle \phi_{\alpha_1},\phi_{\alpha_2},\phi_{\alpha_3}\rangle\rangle^X_{0,\beta}.\end{aligned}$$

Proof. Since all points of $\overline{M}_{0,4}$ are homologous, different points $[D_4]$ determine the same integral series (54). Consider the *boundary* point,

$$[B_4] = \overline{M}_{0,3} \times \overline{M}_{0,3} \subset \overline{M}_{0,5},$$

where $[B_4]$ represents the 4-pointed curve:

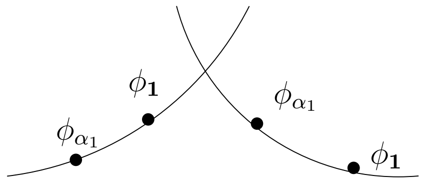

Figure 1

By the splitting axiom of Gromov-Witten theory, the integral series (54) equals:

$$Q^\beta \sum_{\beta_1+\beta_2=\beta} \langle\langle \phi_{\alpha_1},\phi_{\mathbf{1}},\phi_i\rangle\rangle^X_{0,\beta_1}\; g^{ii}\langle\langle \phi_i,\phi_{\alpha_2},\phi_{\mathbf{1}}\rangle\rangle^X_{0,\beta_2},$$

when the class $\nu_4^*([B_4])$ is taken in the integrand. The formula for $\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}(Q,t)$ is then obtained from the fundamental class axiom.

Similarly, $[D_5] \in \overline{M}_{0,5}$ can be specialized to the the *boundary* point,

$$[B_5] = \overline{M}_{0,3} \times \overline{M}_{0,3} \times \overline{M}_{0,3} \subset \overline{M}_{0,5},$$

where $[B_5]$ represents the 5-pointed curve:

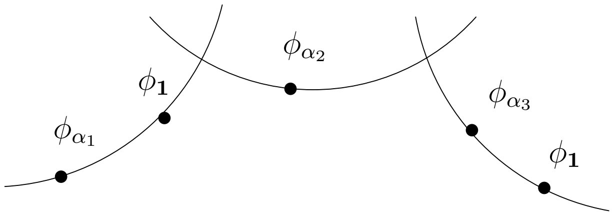

Figure 2

By the splitting axiom of Gromov-Witten theory, the integral series (55) equals:

$$Q^\beta \sum_{\beta_1+\beta_2+\beta_3=\beta} \langle\langle \phi_{\alpha_1}, \phi_{\mathbf{1}}, \phi_i \rangle\rangle^X_{0,\beta_1}\, g^{ii} \langle\langle \phi_i, \phi_{\alpha_2}, \phi_j \rangle\rangle^X_{0,\beta_2}\, g^{jj} \langle\langle \phi_j, \phi_{\alpha_3}, \phi_{\mathbf{1}} \rangle\rangle^X_{0,\beta_3},$$

when the class $\nu_5^*([B_5])$ is taken in the integrand. The formula for $\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{11}}(Q,t)$ is obtained from the fundamental class axiom. $\square$

6.2. Local splitting I. Let $(D_4, x_1, \ldots, x_4)$ be a nonsingular, irreducible 4-pointed curve. Let $[\pi] \in \overline{M}_{0,4+k}(X,\beta)$ represent a moduli point

$$\pi : (C, x_1, \ldots, x_{4+k}) \to X,$$

satisfying $\nu_4([\pi]) = [D_4]$. The curve C may be singular and reducible. However, by the definition of ν_4, there exists a unique *special component* $C' \subset C$ and a canonical isomorphism

$$\nu_4 : C' \stackrel{\sim}{=} D$$

of curves obtained by restriction. Moreover, the first 4 markings of C determine 4 *distinct* points of C' by contraction. The data of C' together with the 4 contracted markings is isomorphic to $(D_4, x_1, \ldots, x_4)$.

We will study the equivariant Gromov-Witten series $\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}(Q,t)$ via localization. Let $\Gamma \in G_{0,4+n}(X,\beta)$ be a graph in the localization sum. Define $\overline{M}_{\Gamma,[D_4]}$ by the intersection:

$$\overline{M}_{\Gamma,[D_4]} = \overline{M}_\Gamma \cap \nu_4^{-1}([D_4]).$$

Let $G^{[D_4]}_{0,4+n}(X,\beta) \subset G_{0,4+n}(X,\beta)$ denote the set of graphs Γ satisfying $\overline{M}_{\Gamma,[D_4]} \neq \emptyset$.

Let $\Gamma \in G^{[D_4]}_{0,4+n}(X,\beta)$. There exists a unique vertex v_Γ of Γ corresponding to the special components C' of the maps

$$[\pi : (C, x_1, \ldots, x_{4+n}) \to X] \in \overline{M}_{\Gamma,[D]}.$$

Let $p_\Gamma = j(v_\Gamma)$ be the $\mathbf{T}$-fixed point of X corresponding to v_Γ. Then, $\pi(C') = p_\Gamma$. Let

$$G^{[D_4],i}_{0,4+n}(X,\beta) \subset G^{[D_4]}_{0,4+n}(X,\beta)$$

denote the set of graphs Γ satisfying $p_\Gamma = p_i$.

We may separate the contributions of graphs Γ to $\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}(Q,t)$ into $m+1$ types corresponding to values of p_Γ:

$$\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}(Q,t) = \sum_{i=0}^{m}\sum_{n\geq 0}\sum_{\Gamma\in G^{[D_4],i}_{0,4+n}(X,\beta)} \mathrm{Cont}_\Gamma(\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}). \tag{56}$$

Lemma 25. *Let p_i be a $\mathbf{T}$-fixed point of X. Then,*

$$\sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G^{[D_4],i}_{0,4+n}(X,\beta)} \mathrm{Cont}_\Gamma(\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}) = \Psi^i_{\alpha_1}\Psi^i_{\alpha_2}.$$

Proof. Let $\Gamma \in G^{[D_4],i}_{0,4+n}(X,\beta)$. The valence of v_Γ in Γ is at least 4. The fixed locus $\overline{M}_\Gamma$ contains the factor $\overline{M}_{0,val(v_\Gamma)}$. In the contribution,

$$\mathrm{Cont}_\Gamma(\theta^\beta_{\alpha_1\alpha_2\mathbf{11}}), \tag{57}$$

the class $\nu_4^*([D_4])$ is pulled back from the factor $\overline{M}_{0,val(v_\Gamma)}$. The contribution (57) is *unchanged* by a different choice of $[D_4] \in \overline{M}_{0,4}$ in the integrand.

We will now replace $[D_4]$ with the boundary point $[B_4] \in \overline{M}_{0,4}$ in the integrand of $\mathrm{Cont}_\Gamma(\theta^\beta_{\alpha_1\alpha_2\mathbf{11}})$. The new choice $[B_4]$ will be used only to analyze the contribution.

As B_4 is reducible, we may apply the splitting formula for the moduli space $\overline{M}_{0,val(v_\Gamma)}$ to the integral $\mathrm{Cont}_\Gamma(\theta^\beta_{\alpha_1\alpha_2\mathbf{11}})$. As the splitting occurs at the vertex v_Γ, Givental describes the resulting equations as *local*.

Let Γ_1 and Γ_2 denote the graphs obtained in a single term of the local splitting of $\mathrm{Cont}_\Gamma(\theta^\beta_{\alpha_1\alpha_2\mathbf{11}})$:

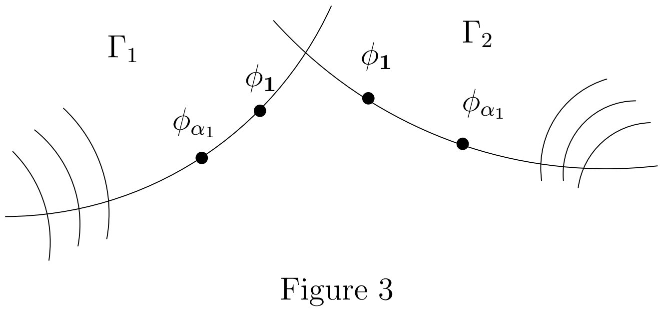

Figure 3

After ordering the extra n_1 markings, the graph Γ_1 lies in the set $G^{\Psi^i_{\alpha_1}}_{0,3+n_1}(X,\beta)$. Similarly, after ordering the extra n_2 markings, the graph Γ_2 lies in $G^{\Psi^i_{\alpha_2}}_{0,3+n_2}(X,\beta)$. Then, a straightforward accounting of graphs and factors in the virtual localization formula completes the proof of the Lemma. $\square$

Lemmas 24 - 25 and equation (56) together imply the first consequence of the local splitting argument.

LEMMA 26.
$$\sum_{i=0}^{m} \Psi^i_{\alpha_1} \Psi^i_{\alpha_2} e_{\alpha_2} = \delta_{\alpha_1\alpha_2}.$$

The matrix Ψ is therefore invertible with inverse determined by
$$\Psi^{-1} = (\Psi g^{-1})^t = g^{-1}\Psi^t.$$

6.3. Local splitting II. Let $(D_5, x_1, \ldots, x_5)$ be a nonsingular, irreducible 5-pointed curve. Let $[\pi] \in \overline{M}_{0,5+k}(X,\beta)$ represent a moduli point,
$$\pi : (C, x_l, \ldots, x_{5+k}) \to X,$$
satisfying $\nu_5([\pi]) = [D_5]$. As before, there exists a unique *special component* $C' \subset C$ and a canonical restriction isomorphism
$$\nu_5 : C' \stackrel{\sim}{=} D.$$
The first 5 markings of C determine 5 *distinct* points of C' by contraction. The data of C' together with the contracted 5 markings is isomorphic to $(D_5, x_1, \ldots, x_5)$.

We will now study the equivariant series $\vartheta^{\beta}_{\alpha_1\alpha_2\alpha_3\mathbf{11}}(q,t)$ via localization. Let $\Gamma \in G_{0,5+n}(X,\beta)$ be a graph in the localization sum. Define $\overline{M}_{\Gamma,[D]}$ by the following intersection:
$$\overline{M}_{\Gamma,[D]} = \overline{M}_{\Gamma} \cap \nu_5^{-1}([D]).$$
Let $G^{[D]}_{0,5+n}(X,\beta) \subset G_{0,5+n}(X,\beta)$ denote the set of graphs Γ satisfying $\overline{M}_{\Gamma,[D]} \neq \emptyset$.

Let $\Gamma \in G^{[D]}_{0,5+n}(X,\beta)$. There exists a unique vertex v_Γ of Γ corresponding to the special components C' of the maps
$$[\pi : (C, x_1, \ldots, x_{5+n}) \to X] \in \overline{M}_{\Gamma,[D]}.$$
Let $p_\Gamma = j(v_\Gamma)$ be the $\mathbf{T}$-fixed point of X corresponding to v_Γ. As before, let
$$G^{[D],i}_{0,5+n}(X,\beta) \subset G^{[D]}_{0,5+n}(X,\beta)$$

denote the set of graphs Γ satisfying $p_\Gamma = p_i$.

We may separate the contributions of graphs Γ to $\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}}(Q,t)$ into $m+1$ types corresponding to values of p_Γ:

$$\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}}(Q,t) = \sum_{i=0}^{m} \sum_{n\geq 0} \sum_{\Gamma \in G^{[D],i}_{0,5+n}(X,\beta)} \mathrm{Cont}_\Gamma(\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}}). \tag{58}$$

Lemma 27. *Let p_i be a $\mathbf{T}$-fixed point of X. Then,*

$$\sum_{n\geq 0} \sum_{\beta \in E} \sum_{\Gamma \in G^{[D_5],i}_{0,5+n}(X,\beta)} \mathrm{Cont}_\Gamma(\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}}) = \Psi^i_{\alpha_1} \frac{\partial \overline{u}^i}{\partial t^{\alpha_3}} \Psi^i_{\alpha_2}.$$

Proof. Let $\Gamma \in G^{[D_5],i}_{0,5+n}(X,\beta)$. The valence of v_Γ in Γ is at least 5. The fixed locus $\overline{M}_\Gamma$ contains the factor $\overline{M}_{0,val(v_\Gamma)}$. In the contribution,

$$\mathrm{Cont}_\Gamma(\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}}), \tag{59}$$

the class $\nu_5^*([D_5])$ is pulled back from the factor $\overline{M}_{0,val(v_\Gamma)}$. The contribution (59) is *unchanged* by a different choice of $[D_5] \in \overline{M}_{0,5}$ in the integrand.

We will now replace $[D_5]$ with the boundary point $[B_5] \in \overline{M}_{0,5}$ in the contribution integral $\mathrm{Cont}_\Gamma(\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}})$. As B_5 is reducible, we may apply the splitting formula for the moduli space $\overline{M}_{0,val(v_\Gamma)}$.

Let Γ_1, Γ_2, and Γ_3 denote the graphs obtained in a single term of the local splitting of $\mathrm{Cont}_\Gamma(\vartheta^\beta_{\alpha_1\alpha_2\alpha_3\mathbf{1}\mathbf{1}})$:

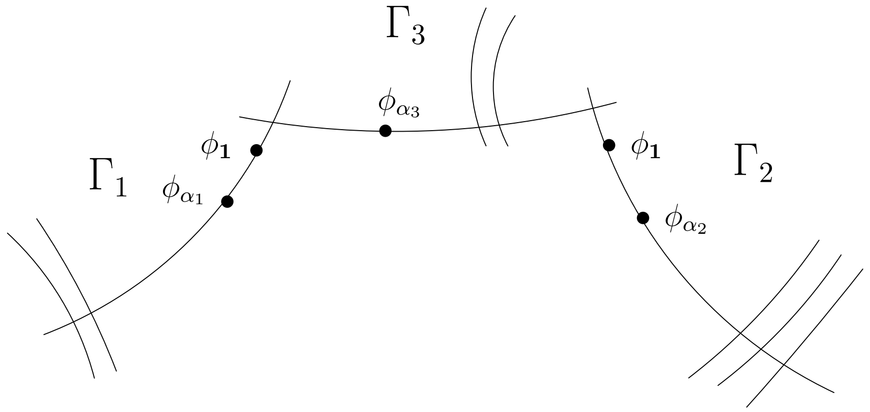

Figure 4

After an ordering of the extra markings n_1, n_2, and n_3, we see

$$\Gamma_1 \in G^{\Psi^i_{\alpha_1}}_{0,3+n_1}(X,\beta_1), \quad \Gamma_2 \in G^{\Psi^i_{\alpha_2}}_{0,3+n_2}(X,\beta_2), \quad \Gamma_3 \in G^{\partial \overline{u}_i/\partial t_{\alpha_3}}_{0,3+n_3}(X,\beta_3).$$

The Lemma then follows by an accounting of all the contributions. $\square$

Lemmas 24, Lemma 27, and equation (58) together imply the second consequence of the local splitting argument.

LEMMA 28.

$$\sum_{i=0}^{m} \Psi^i_{\alpha_1} \frac{\partial \overline{u}^i}{\partial t^{\alpha_3}} \Psi^i_{\alpha_2} e_{\alpha_2} = \frac{\partial^3 F_0^X}{\partial t^{\alpha_1} \partial t^{\alpha_2} \partial t^{\alpha_3}} e_{\alpha_2}.$$

6.4. Proof of Proposition 4 part (i). Let A_ι denote the matrix of multiplication by ∂_ι in the ∂_μ basis,

$$\partial_\iota \cdot \partial_{\mu_2} = \sum_{\mu_1=0}^{m} [A_\iota]^{\mu_1}_{\mu_2} \partial_{\mu_1}.$$

The coefficients of A^t_ι are:

$$[A^t_\iota]^{\mu_2}_{\mu_1} = \frac{\partial^3 F_0^X}{\partial t^{\mu_1} \partial t^{\mu_2} \partial t^{\iota}} e_{\mu_2}.$$

By Lemmas 26 and 28, we see

$$\Psi^t D_\iota (\Psi^{-1})^t = A^t_\iota,$$

$$\Psi^{-1} D_\iota \Psi = A_\iota$$

where D_ι is the diagonal matrix with coefficients $\partial \overline{u}^i / \partial t^\iota$. Hence, Ψ determines a change of basis which simultaneous diagonalizes all the matrices A_ι.

Since $\mathcal{F}^*_{\mathbf{T}}(X)$ is semisimple over $N^*_{\mathbf{T}}(X)[[K^*_{\mathbf{T}}(X)^\vee]]$, simultaneous eigenspaces (of multiplicity 1) of the matrices A_ι are determined by the 1-form sections,

$$\gamma_0, \ldots, \gamma_m,$$

of the formal characteristic subvariety of $\mathcal{F}^*_{\mathbf{T}}(X)$ — see Section 3.3 of Chapter 1. Moreover, the eigenvalues correspond to the coefficients of the 1-forms γ_j in the $dt^0, \ldots, dt^m$ basis.

As X carries a torus action with finitely many fixed points, $H_2(X, \mathbb{Z})$ is torsion free by the Bialynicki-Birula Theorem. $N^*_{\mathbf{T}}(X)$ is then an integral domain by Lemma 21 — applied with ground field $\mathbf{R}^*$ instead of $\mathbb{C}$. Hence, $N^*_{\mathbf{T}}(X)[[K^*_{\mathbf{T}}(X)^\vee]]$ is also an integral domain.

Since the matrices A_ι are defined over the domain $N^*_{\mathbf{T}}(X)[[K^*_{\mathbf{T}}(X)^\vee]]$, their simultaneous eigenspaces and eigenvalues are *uniquely determined* in the multiplicity 1 case. Matching the eigenvalues, we find

$$\frac{\partial \overline{u}^i}{\partial t^\iota} = \frac{\partial u^i}{\partial t^\iota},$$

for all i and ι (the degree 0 terms of the functions $\overline{u}^i$ are used to fix the permutation in the match). Since both $\overline{u}^i(Q,t)$ and $u^i(Q,t)$ vanish at the origin, we conclude

$$\overline{u}^i(Q,t) = u^i(Q,t). \tag{60}$$

The equation $\partial/\partial\overline{u}^i = \epsilon_i$ follows from the identification (60) and the definition $\partial/\partial u^i = \epsilon_i$. $\square$

7. Materialization, norms, and the transition matrix

7.1. Overview. Local splitting equations were used in Section 6 to prove two results:

- Ψ^i_μ is an invertible matrix,
- $\overline{u}^0(Q,t), \ldots, \overline{u}^m(Q,t)$ are canonical coordinates.

The proofs of parts (ii) and (iii) of Proposition 4 require a further study of the transition matrix Ψ^i_μ.

By the invertibility of Ψ^i_μ, independent vector fields $\tilde{\epsilon}_i, \ldots, \tilde{\epsilon}_m$ are defined by the equations

$$\partial_\mu = \sum_{i=0}^{m} \Psi^i_\mu \tilde{\epsilon}_i. \tag{61}$$

Then, the pairing $\langle \partial_\mu, \partial_\nu \rangle = e_\mu^{-1}\delta_{\mu\nu}$ together with Lemma 26 implies:

$$\langle \tilde{\epsilon}_i, \tilde{\epsilon}_j \rangle = \delta_{ij}. \tag{62}$$

Since the vectors $\tilde{\epsilon}_i$ span the simultaneous eigenspaces of the matrices A_ι, each vector $\tilde{\epsilon}_i$ is proportional to ϵ_i. We will prove the identity,

$$\tilde{\epsilon}_i = \sqrt{\Delta^i}\epsilon_i, \tag{63}$$

via local splitting equations.

Equations (62) and (63) together yield part (ii) of Proposition 4,

$$\langle \epsilon_i, \epsilon_i \rangle^{-1} = (\sqrt{\Delta^i})^2.$$

Part (iii) of Proposition 4 is a consequence of definition (61) and the equation (63).

7.2. Preliminaries. We will require a basic consequence of part (i) Proposition 4 in the splitting analysis below.

Lemma 29.

$$\frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}} = 1.$$

Proof. Since $\overline{u}^0, \ldots, \overline{u}^m$ are canonical coordinates and $\partial/\partial t^{\mathbf{1}}$ is the unit field,

$$\frac{\partial}{\partial t^{\mathbf{1}}} = \sum_{i=0}^{m} \frac{\partial}{\partial \overline{u}^i}.$$

The Lemma then follows immediately. $\square$

Following the notation of Section 5, the functions $\partial \overline{u}^i/\partial t^{\mathbf{1}}$ have graph theoretic expressions:

- Let $G^{\partial \overline{u}^i/\partial t^{\mathbf{1}}}_{0,3+n}(X,\beta) \subset G_{0,3+n}(X,\beta)$ denote the set of graphs for which markings 1 and 2 lie on a single irreducible component contracted to p_i. Then,
$$\frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}}(Q,t) = \sum_{n\geq 0} \sum_{\beta \in E} \sum_{\Gamma \in G^{\partial \overline{u}^i/\partial t^{\mathbf{1}}}_{0,3+n}(X,\beta)} \mathrm{Cont}_\Gamma(e_i \frac{\partial^3 F_0^X}{\partial t^i \partial t^i \partial t^{\mathbf{1}}}).$$

While Lemma 29 is a trivial corollary of Proposition 4, a direct derivation from the definition via localization graphs does not appear to be straightforward.

7.3. Local splitting revisited. Let $\Psi^i_{\mathbf{1}} = \sum_{\alpha=0}^m \Psi^i_\alpha$. The first consequence of Lemma 29 is the following identification.

LEMMA 30. $\Psi^i_{\mathbf{1}} = (\sqrt{\Delta^i})^{-1}$.

Proof. We will apply the local splitting technique to deduce:

$$\Psi^i_{\mathbf{1}} \sqrt{\Delta^i} = \frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}} \frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}}. \tag{64}$$

By Lemma 29, the right side of (64) is 1, hence $\Psi^i_{\mathbf{1}} = (\sqrt{\Delta^i})^{-1}$.

The proof of equation (64) requires a new summation of localization graphs $\sigma^i(Q,t)$:

- Let $G^{\sigma^i}_{0,4+n}(X,\beta) \subset G_{0,4+n}(X,\beta)$ denote the set of graphs for which markings 1 and 2 lie on an irreducible component E contracted to p_i *and* markings 3 and 4 lie on *distinct* branches off of E. Let
$$\sigma^i(Q,t) = \sum_{n\geq 0} \sum_{\beta \in E} \sum_{\Gamma \in G^{\sigma^i}_{0,4+n}(X,\beta)} \mathrm{Cont}_\Gamma(\theta^\beta_{ii\mathbf{11}}).$$

  The series $\theta^\beta_{ii\mathbf{11}}$ is defined by (54).

Let B_4, B_4' be the boundary moduli points of $\overline{M}_{0,4}$ defined by:

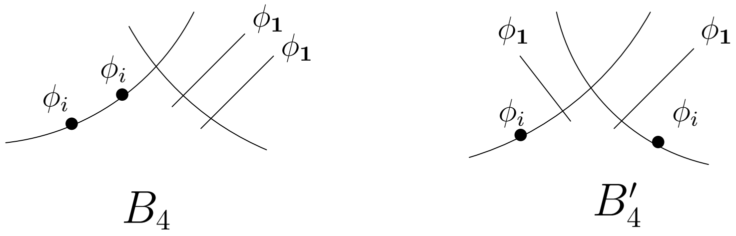

Figure 5

By specializing the curve D_4 in the integrand of $\theta^{\beta}_{ii\mathbf{1}\mathbf{1}}$ to B_4 and applying the local splitting equation, we find:

$$\sigma^i = \Psi^i_{\mathbf{1}} \frac{1}{e_i} \sqrt{\Delta^i}.$$

Similarly, specialization to B_4' yields:

$$\sigma^i = \frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}} \frac{1}{e_i} \frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}},$$

completing the proof of equation (64). □

The last Lemma needed in the proof of Proposition 4 is proven by the same method used in Lemma 30.

Lemma 31. $\Psi^i_\mu = \Psi^i_{\mathbf{1}}\ \partial \overline{u}^i / \partial t^\mu$.

Proof. We will apply the local splitting technique to deduce the equation

$$\Psi^i_\mu \frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}} = \Psi^i_{\mathbf{1}} \frac{\partial \overline{u}^i}{\partial t^\mu}. \tag{65}$$

The Lemma is then a consequence of Lemma 29.

We will require a summation of localization graphs $\tau^i_\mu(Q,t)$ for the derivation of equation (65):

- Let $G^{\tau^i_\mu}_{0,4+n}(X,\beta) \subset G_{0,4+n}(X,\beta)$ denote the set of graphs for which the marking 1 lies on an irreducible component E contracted to p_i *and* markings 2,3, and 4 lie on *distinct* branches off of E. Then,

$$\tau^i_\mu(Q,t) = \sum_{n\geq 0} \sum_{\beta \in E} \sum_{\Gamma \in G^{\tau^i_\mu}_{0,4+n}(X,\beta)} \mathrm{Cont}_\Gamma(\theta^{\beta}_{i\mu\mathbf{1}\mathbf{1}}).$$

Let B_4, B_4' be the boundary moduli points of $\overline{M}_{0,4}$ defined by:

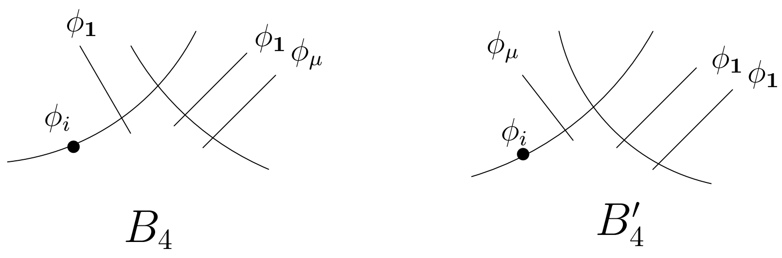

Figure 6

By specializing the curve D_4 in the integrand of $\theta^\beta_{i\mu\mathbf{1}\mathbf{1}}$ to B_4 and applying the local splitting equation, we find:

$$\tau^i_\mu = \Psi^i_\mu \frac{1}{e_i} \frac{\partial \overline{u}^i}{\partial t^{\mathbf{1}}}.$$

Similarly, specialization to B_4' yields:

$$\tau^i_\mu = \Psi^i_{\mathbf{1}} \frac{1}{e_i} \frac{\partial \overline{u}^i}{\partial t^\mu},$$

completing the proof of equation (65). $\square$

7.4. Completion of the proof of Proposition 4. Lemmas 30 and 31 together yield the equation:

$$\Psi^i_\mu = (\sqrt{\Delta^i})^{-1} \frac{\partial \overline{u}^i}{\partial t^\mu}.$$

Then, the definitions

$$\partial_\mu = \sum_{i=0}^m \Psi^i_\mu \tilde{\epsilon}_i, \quad \frac{\partial}{\partial \overline{u}_i} = \epsilon_i$$

imply $\tilde{\epsilon}_i = \sqrt{\Delta^i}\epsilon_i$. $\square$

CHAPTER 5

Quantum potentials, the string flow, and the dilaton flow

1. Overview

We study here integrals over the moduli spaces $\overline{M}_{0,n}$ which will be required for our analysis of equivariant Gromov-Witten theory in Chapter 7. We prove the integral series which arise are all functions of a distinguished series $u(T)$. The series $u(T)$ plays an important role: $u(T)$ arises as the principal vertex integral in the materialization of canonical coordinates and as the parameter of string flow.

2. Quantum potentials in genus 0

Let $S_i(\psi_i)$ be a formal series in the variable ψ_i:

$$S_i(\psi_i) = S_{i0}\psi_i^0 + S_{i1}\psi_i^1 + S_{i2}\psi_i^2 + \dots\ ,$$

where S_{ij} are independent variables. The *empty* superscript after the brackets $\langle , \rangle$ will be used for integrals of the ψ classes over the moduli of curves:

$$\langle S_1(\psi_1), \dots, S_n(\psi_n)\rangle_{0,n} = \int_{\overline{M}_{0,n}} \prod_{i=1}^{n} S_i(\psi_i).$$

On the right side, the integrals over moduli space are viewed as multilinear in the formal parameters S_{ij}.

Let T denote the following set of formal variables:

$$T = \{T_0, T_1, T_2, \dots\}.$$

Let $T(\psi)$ denote the formal series:

$$T(\psi) = T_0\psi^0 + T_1\psi^1 + T_2\psi^2 + \dots\ .$$

When $T(\psi)$ occurs in the position of the i^{th} point in a generating series, the substitution $T(\psi = \psi_i)$ will be understood.

The study of higher genus structures requires an analysis of several quantum potentials. Let $F_0(T)$ denote the standard genus 0 potential:

$$F_0(T) = \sum_{n=3}^{\infty} \frac{1}{n!} \langle T(\psi), \cdots, T(\psi) \rangle_{0,n}.$$

By definition, $F_0(T) \in \mathbb{Q}[[T]]$. However, a stronger result holds. Let $T_{\geq 2} = \{T_2, T_3, T_4, \ldots\}$ be the restricted variable set.

Lemma 32. $F_0(T) \in \mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}][[T_0]]$.

Proof. The T_1 dependence of $F_0(T)$ is determined by the *dilaton equation*,

$$\int_{\overline{M}_{0,n}} \prod_{i=1}^{n} \psi_i^{a_i} \cdot \psi_{n+1} = (-2+n) \int_{\overline{M}_{0,n}} \prod_{i=1}^{n} \psi_i^{a_i}.$$

Using the dilaton equation, the potential $F_0(T)$ can be expressed as:

$$F_0(T) = \sum_{i \geq 3} \frac{T_0^i}{i!} \sum_{\mathbf{a}} \frac{1}{l(\mathbf{a})!} \frac{\prod_{j=1}^{l(\mathbf{a})} T_{a_j}}{(1-T_1)^{l(\mathbf{a})+i-2}} \int_{\overline{M}_{0,l(\mathbf{a})+i}} \prod_{j=1}^{l(\mathbf{a})} \psi_j^{a_j}. \tag{66}$$

The second sum is over all finite ordered sequences $\mathbf{a}$ of length $l(\mathbf{a}) \geq 0$,

$$\mathbf{a} = (a_1, \ldots, a_{l(\mathbf{a})})$$

satisfying $a_j \geq 2$.

The integral over $\overline{M}_{0,l(\mathbf{a})+i}$ in (66) vanishes unless

$$\sum_{j=1}^{l(\mathbf{a})} a_j = \dim_{\mathbb{C}} \overline{M}_{0,l(\mathbf{a})+i} = l(\mathbf{a}) + i - 3.$$

For each $i \geq 3$, there are only finitely many sequences $\mathbf{a}$ which satisfy the above dimension constraint. Hence, for each i, the second sum in (66) defines a polynomial in the variables $T_{\geq 2}$ and $(1-T_1)^{-1}$. $\square$

Let $u(T)$ and $\sqrt{\Delta}(T)$ be defined by differentiating $F_0(T)$,

$$u(T) = \frac{\partial^2 F_0}{\partial T_0^2}(T) \in \mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}][[T_0]],$$

$$\sqrt{\Delta}(T) = \frac{\partial^3 F_0}{\partial T_0^3}(T) \in \mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}][[T_0]].$$

Equivalently,

$$u(T) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle 1, 1, T(\psi), \cdots, T(\psi) \rangle_{0,n+2}. \tag{67}$$

$$\sqrt{\Delta}(T) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle 1, 1, 1, T(\psi), \cdots, T(\psi) \rangle_{0,n+3}. \tag{68}$$

The restrictions of the series to the hypersurface $T_0 = 0$ are obtained from their defining integrals using dimension constraints:

$$u(T)|_{T_0=0} = 0, \quad \sqrt{\Delta}(T)|_{T_0=0} = \frac{1}{1-T_1}.$$

A connection between $u(T), \sqrt{\Delta}(T)$ and the series $u^i(Q,t), \sqrt{\Delta^i}(Q,t)$ arising in the materialization of canonical coordinates will be derived in Chapter 7.

Two other potentials will play basic roles in the localization analysis in Chapter 7:

$$v_1(T,x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \langle 1, \frac{1}{x-\psi}, T(\psi), \cdots, T(\psi) \rangle_{0,n+2},$$

$$v_2(T,x,y) = \frac{1}{x+y} + \sum_{n=1}^{\infty} \frac{1}{n!} \langle \frac{1}{x-\psi}, \frac{1}{y-\psi}, T(\psi), \cdots, T(\psi) \rangle_{0,n+2}.$$

After extracting the T_1 dependence via the dilaton equation (as in the proof of Lemma 32), we find:

$$v_1(T,x) \in \mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}, \frac{1}{x}][[T_0]], \tag{69}$$

$$v_2(T,x,y) \in \mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}, \frac{1}{x}, \frac{1}{y}, \frac{1}{x+y}][[T_0]].$$

The series $u(T)$ determines all three of the series $\sqrt{\Delta}(T)$, $v_1(T,x)$ and $v_2(T,x,y)$ by the main result of Chapter 5.

Proposition 5. *The following relations hold:*

$$\begin{aligned} \sqrt{\Delta}(T) &= \frac{1}{1 - \sum_{n\geq 0} T_{n+1} \frac{u^n(T)}{n!}}, \\ v_1(T,x) &= e^{u(T)/x}, \\ v_2(T,x,y) &= \frac{e^{u(T)(\frac{1}{x}+\frac{1}{y})}}{x+y}. \end{aligned}$$

The proof of Proposition 5 will be given in Section 4 after a discussion of the string flow.

3. The string flow

Let $\partial_i = \partial / \partial T_i$. Define the string operator by:

$$\mathcal{L} = \partial_0 - \sum_{i=0}^{\infty} T_{i+1} \partial_i .$$

$\mathcal{L}$ may be viewed as a vector field on the infinite dimensional manifold $\mathcal{T}$ with coordinates T.

Let $p \in \mathcal{T}$ be a point with coordinates $T_i(p) = p_i$. Consider the formal path $\gamma(\tau)$ in $\mathcal{T}$ with parameter τ defined by the following equations:

$$\begin{aligned} T_0(\gamma(\tau)) &= \tau + \sum_{n=0}^{\infty} (-1)^n p_n \frac{\tau^n}{n!}, \\ T_1(\gamma(\tau)) &= 1 - \frac{dT_0(\tau)}{d\tau}, \\ T_i(\gamma(\tau)) &= -\frac{dT_{i-1}(\tau)}{d\tau}, \quad i \geq 2, \end{aligned} \tag{70}$$

where $T_i(\gamma(\tau)) \in \mathbb{C}[[\tau]]$.

Lemma 33. *The path γ determines a formal integral curve of $\mathcal{L}$ at p.*

Proof. The evaluations $\tau_i(\gamma(0)) = p_i$ for $i \geq 0$ follow directly from the definition of the formal path (70). Hence $\gamma(0) = p$.

To prove γ is an integral curve of $\mathcal{L}$, we must show

$$\frac{d}{d\tau}\gamma = \mathcal{L}(\gamma). \tag{71}$$

A direct calculation using the definitions of γ and $\mathcal{L}$ verifies (71). □

The integral curve γ defines the formal *string flow* at p in the manifold $\mathcal{T}$.

4. Proof of Proposition 5

The string equation in genus 0,

$$\int_{\overline{M}_{0,n+1}} \prod_{i=1}^{n} \psi_i^{a_i} \cdot \psi_{n+1}^{0} = \sum_{i=1}^{n} \int_{\overline{M}_{0,n}} \psi_i^{a_i - 1} \cdot \prod_{j \neq i} \psi_j^{a_j},$$

is equivalent to the equation:

$$\mathcal{L} F_0 = \frac{T_0^2}{2}. \tag{72}$$

The action of the string operator on the series $u(T)$, $\sqrt{\Delta}(T)$, $v_1(T,z)$, and $v_2(T,x,y)$ is determined by the following Lemma.

Lemma 34. *We have,*

$$\begin{aligned} \mathcal{L}u &= 1, \\ \mathcal{L}\sqrt{\Delta} &= 0, \\ \mathcal{L}v_1 &= \frac{v_1}{z}, \\ \mathcal{L}v_2 &= (\frac{1}{x}+\frac{1}{y})v_2. \end{aligned}$$

Proof. The functions $\mathcal{L}u$, $\mathcal{L}\sqrt{\Delta}$, $\mathcal{L}v_1$, and $\mathcal{L}v_2$ can be directly computed via the string equation from the definitions of the four series. An alternative computation of $\mathcal{L}u$ may be obtained as follows. Since $\mathcal{L}$ and ∂_0 commute, we find

$$\mathcal{L}u = \mathcal{L}\partial_0^2 F_0 = \partial_0^2 \mathcal{L}F_0 = \partial_0^2(\frac{T_0^2}{2}) = 1,$$

using (72). $\square$

We now prove the first equality of Proposition 5:

$$\sqrt{\Delta}(T) = \frac{1}{1-\sum_{n\geq 0} T_{n+1}\frac{u^n(T)}{n!}} \quad . \tag{73}$$

Both sides of (73) specialize to $\frac{1}{1-T_1}$ when restricted to the hypersurface $T_0=0$. The function $\sqrt{\Delta}(T)$ is annihilated by $\mathcal{L}$ by Lemma 34. A direct computation using $\mathcal{L}u=1$ shows the right side of (73) is also annihilated by $\mathcal{L}$. Since series in $\mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}][[T_0]]$ annihilated by $\mathcal{L}$ are determined by their restrictions to $T_0=0$, equation (73) is proven.

Let $p\in\mathcal{T}$ be a point satisfying $p_0=0$ and $p_1\neq 1$. Let $\gamma(\tau)$ be the string flow at p defined in Section 3.

Lemma 35. *The series u is the parameter of the string flow:*

$$u(\gamma(\tau)) = \tau.$$

Proof. Since $u(T)\in\mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}][[T_0]]$ and,

$$T_0(\gamma(0)) = p_0 = 0, \quad T_1(\gamma(0)) = p_1 \neq 1,$$

the composition $u(\gamma(\tau))$ is a well-defined formal series in τ. Moreover, since $u(T)$ is divisible by T_0, $u(\gamma(0))=0$. A differential equation is obtained from Lemmas 33-34,

$$\frac{d}{d\tau}u(\gamma(\tau)) = \mathcal{L}u(\gamma(\tau)) = 1.$$

With the initial condition $u(\gamma(0)) = 0$, we conclude $u(\gamma(\tau)) = \tau$. $\square$

Lemma 36. *We have,*

$$\begin{aligned}
\sqrt{\Delta}(\gamma(\tau)) &= \frac{1}{1-p_1}, \\
v_1(\gamma(\tau), x) &= e^{\tau/x}, \\
v_2(\gamma(\tau), x, y) &= \frac{e^{\tau(\frac{1}{x}+\frac{1}{y})}}{x+y}.
\end{aligned}$$

Proof. The compositions $\sqrt{\Delta}(\gamma(\tau))$, $v_1(\gamma(\tau), x)$ and $v_2(\gamma(\tau), x, y)$ are well-defined formal series in τ by (69) since $p_0 = 0$ and $p_1 \neq 1$. The initial conditions,

$$\begin{aligned}
\sqrt{\Delta}(\gamma(0)) &= \frac{1}{1-p_1}, \\
v_1(\gamma(0), z) &= 1, \\
v_2(\gamma(0), x, y) &= \frac{1}{x+y},
\end{aligned}$$

are determined from the definitions. The Lemma is then proven by the differential equations obtained from Lemmas 33-34,

$$\begin{aligned}
\frac{d}{d\tau}\sqrt{\Delta}(\gamma(\tau)) &= 0, \\
\frac{d}{d\tau}v_1(\gamma(\tau), x) &= \frac{v_1(\gamma(\tau), x)}{x}, \\
\frac{d}{d\tau}v_2(\gamma(\tau), x, y) &= (\frac{1}{x}+\frac{1}{y})v_2(\gamma(\tau), x, y),
\end{aligned}$$

and the uniqueness result for their solutions. $\square$

We now complete the proof of Proposition 5. First, consider the two series:

$$v_1(T, x),\ e^{u(T)/x} \ \in\ \mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}, \frac{1}{x}][[T_0]].$$

To establish the relation $v_1(T, x) = e^{u(T)/x}$, we will prove the equality of derivatives,

$$\frac{\partial^k}{\partial T_0^k} v_1(T, x)|_{T_0=0} = \frac{\partial^k}{\partial T_0^k} e^{u(T)/x}|_{T_0=0}, \tag{74}$$

for all k.

Since both sides of (74) are polynomials in the ring

$$\mathbb{Q}[T_{\geq 2}, \frac{1}{1-T_1}, \frac{1}{x}],$$

it suffices to prove the equality (74) holds after evaluation at all points $p \in \mathcal{T}$ satisfying $p_0 = 0$ and $p_1 \neq 1$.

By Lemmas 33 and 36, the equalities,

$$\mathcal{L}^k v_1(T,x)|_p = \mathcal{L}^k e^{u(T)/x}|_p,$$

hold for all k and all points $p \in \mathcal{T}$ satisfying $p_0 = 0$ and $p_1 \neq 1$. The relation,

$$\partial_0 = \frac{1}{1-T_1}\mathcal{L} + \frac{1}{1-T_1}\sum_{i\geq 1} T_{i+1}\partial_i,$$

then implies equation (74) holds for all k after evaluation at p.

We have proven $v_1(T,x) = e^{u(T)/x}$. The proof of the equality

$$v_2(T,x,y) = \frac{e^{u(T)(\frac{1}{x}+\frac{1}{y})}}{x+y}.$$

is identical. $\square$

5. The dilaton flow

Define the dilaton operator by:

$$\mathcal{D} = \partial_1 - \sum_0^\infty T_i\partial_i.$$

The *dilaton flow* γ' at a point $p \in \mathcal{T}$ is defined by:

$$\begin{aligned} T_1(\gamma'(\tau)) =& 1-(1-p_1)e^{-\tau}, \\ T_i(\gamma'(\tau)) =& e^{-\tau}p_i, \qquad\qquad i \neq 1 \end{aligned} \tag{75}$$

A direct verification shows γ' is an integral curve of $\mathcal{D}$ on $\mathcal{T}$.

6. The string and dilaton flows

Let $p \in \mathcal{T}$. We will now use the string and dilaton flows to canonically associate a point $r(p) \in \mathcal{T}$ satisfying $r_0 = r_1 = 0$.

We first follow the string flow γ_p for time $-u(p)$. The result is the point $q = \gamma_p(-u(p))$ with coordinates well-defined in $\mathbb{Q}[p_{\geq 2}, \frac{1}{1-p_1}][[p_0]]$.

Lemma 37. $q_0 = 0$.

Proof. By the formula for q_0, we must prove the vanishing of

$$-u(p) + \sum_{n=0} p_n \frac{u^n(p)}{n!}.$$

Since the above series vanishes along the hypersurface $p_0 = 0$ and is annihilated by $\mathcal{L}$, the series vanishes identically. $\square$

LEMMA 38. $\frac{1}{1-q_1} = \sqrt{\Delta}(p)$.

Proof. The result is obtained from the formula for q_1 together with Proposition 5. Alternatively, the result may be obtained by observing that $\sqrt{\Delta}$ is invariant under the string flow and $\sqrt{\Delta}(q) = \frac{1}{1-q_1}$. $\square$

Next, we follow the dilaton flow for time $-\log\sqrt{\Delta}(p)$. By the formulas for the dilaton flow, the point $r = \gamma'_q(-\log\sqrt{\Delta}(p))$ satisfies $r_0 = r_1 = 0$. The coordinates of r are well-defined in $\mathbb{Q}[p_{\geq 2}, \frac{1}{1-p_1}][[p_0]]$.

CHAPTER 6

Givental's formula for higher genus potentials

1. Genus 0 potentials

Let $\mathcal{F}$ be a Frobenius manifold, and let p be a semisimple point. We will study $\mathcal{F}$ near p, either on a open set U if $\mathcal{F}$ is a complex Frobenius manifold, or on a formal neighborhood, if $\mathcal{F}$ is formal.

Let Φ be a potential function for $\mathcal{F}$ at p obtained from the potential condition. The potential Φ is uniquely specified modulo quadratic terms. The genus 0 potential $F_0^{\mathcal{F}}$ of $\mathcal{F}$ is *defined* by

$$F_0^{\mathcal{F}} = \Phi, \tag{76}$$

well-defined modulo quadratic terms.

For Frobenius manifolds $\mathcal{F}(X)$ obtained from Gromov-Witten theory, the potential Φ is defined to equal the genus 0 Gromov-Witten potential of X. Definition (76) is motivated by Gromov-Witten theory.

2. R-Calibrations

For the definitions of the potentials in genus $g \geq 1$, we will require further data. Let p be an R-calibrated semisimple point of $\mathcal{F}$. Following the notation of Section 4.5 of Chapter 1, an R-calibration consists of a selection of square roots $\sqrt{\Delta^j}$, where

$$\Delta^j = \langle \epsilon_j, \epsilon_j \rangle^{-1},$$

and a selection of a formal fundamental solution in canonical coordinates,

$$\tilde{S} = R(z, u) e^{\mathbf{u}/z},$$

satisfying properties (i-ii) of Theorem 1. If $\mathcal{F}$ is conformal, the unique fundamental solution satisfying property (iv) is selected.

3. Genus 1 potentials

The differential $dF_1^{\mathcal{F}}$ of the genus 1 potential of $\mathcal{F}$ is defined at p by

$$dF_1^{\mathcal{F}} = \sum_i \frac{1}{48} d(\log \Delta^i) + \frac{1}{2}(R_1)_i^i du^i, \tag{77}$$

where the index i parameterizes the canonical coordinates at p.

The right side of the definition is a closed 1-form by the following calculation:

$$\begin{aligned} d\sum_i \frac{1}{48} d(\log \Delta^i) + \frac{1}{2}(R_1)^i_i du^i &= \sum_i \frac{1}{2} d(R_1)^i_i \wedge du^i \\ &= \sum_{i,k} \frac{1}{2}(du^k - du^i)(R_1)^i_k (R_1)^k_i \wedge du^i \\ &= \sum_{i,k} \frac{1}{2}(R_1)^i_k (R_1)^k_i du^k \wedge du^i \\ &= 0, \end{aligned}$$

where equation (24) is used to deduce the second equality. Hence, the right side of (77) is locally exact, and F_1 is well-defined up to a constant.

4. Higher genus potentials

The potential $F^{\mathcal{F}}_g$ for $g \geq 2$ associated to an R-calibrated Frobenius manifold $\mathcal{F}$ is defined by the following formula due to Givental:

$$\begin{aligned} &\exp\left(\sum_{g\geq 2} \lambda^{g-1} F^{\mathcal{F}}_g\right) = \\ (78)\qquad &\left[\exp\Big(\frac{\lambda}{2}\sum_{k,l=0}^{\infty}\sum_{i,j} E^{ij}_{kl}\sqrt{\Delta^i}\sqrt{\Delta^j}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}\Big)\prod_m \tau(\lambda\Delta^m, Q^m)\right]_{Q^i_k = T^i_k}, \end{aligned}$$

The variables and functions on the right side of the formula all require definitions. The indices i, j, and m parameterize the canonical coordinates at p. For each canonical coordinate u^m, Q^m denotes the variable set $\{Q^m_k\}_{k\geq 0}$. The functions E^{ij}_{kl}, T^i_k, and τ are defined below. The first two are obtained from the R-calibration and the last is the τ-function of the moduli space of curves.

Let S be the formal fundamental solution,

$$S = \Psi^{-1} R(z, u) e^{\mathbf{u}/z}, \tag{79}$$

obtained from the R-calibration for a choice of flat coordinates. The notation $S(z)$ will be used below to make the z dependence explicit.

The functions E^{ij}_{kl} and T^i_k are defined near p. We first define E^{ij}_{kl}. Define the matrix $E(w, z)$ by:

$$E^{ij}(w, z) = \frac{1}{w+z}\sum_{\mu,\nu} (S^{\dagger}(w))^i_{\mu} g_{\mu\nu} (S(z))^{\nu}_j. \tag{80}$$

Using (79) and the relation

$$(\Psi^{-1})^t g \Psi^{-1} = 1,$$

we find,

$$E^{ij} = \frac{e^{u^i/w + u^j/z}}{w+z} \sum_m R_i^m(w) R_j^m(z).$$

Then, by the unitary property (ii) of the R-calibration,

$$\frac{1}{w+z} \sum_m R_i^m(w) R_j^m(z)$$

is a power series in z and w except for an initial pole. The functions E_{kl}^{ij} are defined as coefficients of the expansion:

$$E^{ij}(w,z) = e^{u^i/w + u^j/z} \left(\frac{\delta_{ij}}{w+z} + \sum_{k,l=0}^{\infty} E_{kl}^{ij} (-w)^k (-z)^l \right). \tag{81}$$

While the product on the right side of (81) is formal, the functions E_{kl}^{ij} are well-defined (and independent of the flat coordinate choice).

To define T_k^i, we first consider the identity $\mathbf{1}$ of the Frobenius manifold in flat coordinates:

$$\mathbf{1} = \sum_i \epsilon_i = \sum_\mu \delta^\mu \partial_\mu.$$

We then expand $\sum_\mu \delta_\mu S_i^\mu(z)$ in power series in z:

$$\begin{aligned} S_i^{\mathbf{1}}(z) &= \sum_\mu \delta_\mu S_i^\mu(z) \\ &= e^{u^i/z} \sum_j \frac{1}{\sqrt{\Delta^j}} R_i^j(z), \\ &= \frac{e^{u^i/z}}{\sqrt{\Delta^i}} \left(1 - \sum_{k=2}^{\infty} T_k^i (-z)^{k-1} \right), \end{aligned}$$

where the functions T_k^i are defined by the last equality (together with the conditions $T_0^i = T_1^i = 0$). Again, although the products above are formal, the functions T_k^i are well-defined.

Let Q denote the variable set $\{Q_k\}_{k \geq 0}$. Let

$$Q(\psi) = \sum_{k=0}^{\infty} Q_k \psi^k.$$

The function $\tau(\lambda, Q)$ is the τ-function of the moduli space of curves:

$$\tau(\lambda, Q) = \exp\left(\sum_{g=0}^{\infty} \lambda^{g-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\overline{M}_{g,n}} Q(\psi_1) \ldots Q(\psi_n)\right),$$

where the unstable moduli spaces are omitted as usual. The function $\tau(\lambda, Q)$ is independent of $\mathcal{F}$.

The right side of (78) is now well-defined by the above conventions. Since the evaluations of $\tau(\lambda\delta_m, Q^m)$ occur at $Q_0^m = Q_1^m = 0$, the right side contains no terms with λ exponents less than 1. Hence, the logarithm of the right side may be written as

$$\exp\left(\sum_{g\geq 2} \lambda^{g-1} F_g^{\mathcal{F}}\right),$$

where $F_g^{\mathcal{F}}$ is the genus g *Givental potential* associated to $\mathcal{F}$ at an R-calibrated semisimple point.

5. The conformal case

Let $\mathcal{F}$ be a conformal Frobenius manifold. A canonical R-calibration at p exists up to the choice of square roots $\sqrt{\Delta^i}$ by part (iv) of Theorem 1.

Lemma 39. *The higher genus potentials $F_{g\geq 1}$ for $\mathcal{F}$ are independent of the square root choice.*

Proof. Let $\sqrt{\Delta^i}$ and $\sqrt{\Delta^{i'}}$ be two choices of square roots. Let R be the homogeneous matrix series solution for $\sqrt{\Delta^i}$. The matrix solution R' for $\sqrt{\Delta^{i'}}$ is

$$R' = DRD,$$

where D is the diagonal matrix with coefficients $\sqrt{\Delta^i}/\sqrt{\Delta^{i'}}$, see the proof of Lemma 15.

Since the diagonal elements of R' and R agree,

$$(R'_1)_i^i = (R_1)_i^i,$$

the definition of the genus 1 potential is independent of the square root choice.

By the formulas defining E_{kl}^{ij} and T_k^i, we see,

$$(E')_{kl}^{ij} = E_{kl}^{ij} \frac{\sqrt{\Delta^i}}{\sqrt{\Delta^{i'}}} \frac{\sqrt{\Delta^j}}{\sqrt{\Delta^{j'}}},$$

$$(T')_k^i = T_k^i.$$

Hence, formula (78) is also independent of the square root choice. $\square$

Therefore, *canonical* higher genus potentials are defined for conformal Frobenius manifolds $\mathcal{F}$ at semisimple points.

Formal Frobenius manifolds $\mathcal{F}(X)$ obtained from Gromov-Witten theory carry natural higher genus potentials defined by geometry:

$$F_g^X = \sum_{n,\beta} \frac{Q^\beta}{n!} \langle \gamma, \ldots, \gamma \rangle_{g,n,\beta}^X \tag{82}$$

where $\gamma = \sum_\mu t^\mu T_\mu$. These potentials encode the higher genus Gromov-Witten invariants of X.

We now assume that $\mathcal{F}(X)$ is a complex semisimple Frobenius manifold near the origin. Since $\mathcal{F}(X)$ is conformal, we have *two* definitions of the potentials in genus $g \geq 1$. The first is the Givental potential obtained from the R-calibration via (77) and (78), and the second is the Gromov-Witten potential obtained from geometry (82).

Givental's Conjecture. *The Givental and Gromov-Witten potentials of $\mathcal{F}(X)$ are equal at every semisimple point of $\mathcal{F}(X)$.*

If true, the conjecture implies all the higher genus Gromov-Witten invariants of X in the semisimple case are determined by the genus 0 invariants together with the τ function of the moduli space of curves.

Using Getzler's descendent relation in $H^4(\overline{M}_{1,4}, \mathbb{Q})$, Dubrovin and Zhang have proven Givental's conjecture in genus 1. In fact, Givental's conjecture in genus 1 and the proof by Dubrovin and Zhang occurred simultanenously. For all genera $g \geq 2$, the conjecture remains open. Givental's proof of the conjecture for $\mathbf{P}^n$ via equivariant Gromov-Witten theory will be explained in Part 3.

Only *primary field* potentials have been discussed here. In fact, higher genus descendent potentials have been defined by Givental for calibrated semisimple Frobenius manifolds. Formulas for the descendent potentials and the descendent generalization of Givental's conjecture will be discussed in Part 2.

CHAPTER 7

Givental's formula in equivariant Gromov–Witten theory

1. Overview

Let X be a nonsingular projective variety with a $\mathbf{T}$-action satisfying conditions (i-ii) of Chapter 3. We consider here the formal Frobenius manifold $\mathcal{F}^*_{\mathbf{T}}(X)$ obtained from the localized equivariant Gromov-Witten theory of X. Since $\mathcal{F}^*_{\mathbf{T}}(X)$ is semisimple at the origin by Lemma 23, each R-calibration determines a higher genus potential function by (77) and (78). An important result proven by Givental is an equivariant analogue of his potential conjecture: the Givental potential of $\mathcal{F}^*_{\mathbf{T}}(X)$ *equals* the equivariant Gromov-Witten potential for a distinguished R-calibration.

First a canonical R-calibration of $\mathcal{F}^*_{\mathbf{T}}(X)$ obtained from Gromov-Witten theory will be defined. The R-calibration required for Givental's conjecture is obtained by a distinguished modification of the canonical calibration.

2. The functions A^i

We will require several functions defined by sums over localization graphs following the notation of Chapter 4.

For each fixed point $p_i \in X$, the function A^i is defined as a sum over graphs for which the first marking lies at an end of the graph over p_i:

- Let $G^{A^i}_{0,1+n}(X,\beta) \subset G_{0,1+n}(X,\beta)$ denote the set of graphs for which the marking 1 lies on a valence two vertex mapped to p_i. Then,

$$A^i(Q,t,\psi) = t^i + \sum_{n\geq 0}\sum_{\beta\in E}\sum_{\Gamma\in G^{A^i}_{0,1+n}(X,\beta)} \frac{1}{\chi_\Gamma - \psi}\frac{\mathrm{Cont}_\Gamma\ e_i\langle\phi_i,\gamma^n\rangle}{n!},$$

  where χ_Γ is the torus character corresponding to the unique edge Γ incident to the marking 1, $\gamma = \sum_\mu t^\mu\phi_\mu$, and γ^n is shorthand for n insertions of γ.

The valence two condition implies that the marking lies at an end of the graph. In particular, $G^{A^i}_{0,1+n}(X,\beta)$ contains no degree 0 graphs. The leading term, t^i, may be viewed as a degenerate degree 0 contribution. We may expand A_i as a series in the variable ψ,

$$A^i(\psi) = \sum_{k\geq 0} A^i_k \psi^k,$$

where A^i_k are functions of Q, t.

Recall the functions $u(T)$ and $\sqrt{\Delta}(T)$ defined in Chapter 5. After substitution, $u(A^i)$ and $\sqrt{\Delta}(A^i)$ are functions of Q, t.

Lemma 40. *$u(A^i) = u^i(Q,t)$, $\sqrt{e_i}\sqrt{\Delta}(A^i) = \sqrt{\Delta^i}(Q,t)$.*

Proof. The Lemma is a direct consequence of the materialization formula for $u^i(Q,t)$ and $\sqrt{\Delta^i}(Q,t)$, the formula for graph contributions, and the definitions of $u(T)$, $\sqrt{\Delta}(T)$, and A^i. □

3. The canonical calibration

Canonical coordinates $u^i(Q,t)$ at the origin for the Frobenius manifold $\mathcal{F}^*_{\mathbf{T}}(X)$ are defined by the materialization graph sums of Proposition 4. To define an R-calibration for $\mathcal{F}^*_{\mathbf{T}}(X)$ at the origin, a selection of square roots $\sqrt{\Delta^i}$ must be made. We choose the canonical square roots,

$$\sqrt{\Delta^i},$$

defined by materialization. The fundamental solution required for the R-calibration is given by the following Lemma.

Lemma 41. *The matrix series,*

$$S^\mu_j = \delta^\mu_j \sqrt{e_j} + \langle\langle \phi_\mu, \frac{\phi_j}{z-\psi}\rangle\rangle^X_0 e_\mu \sqrt{e_j}, \tag{83}$$

factors canonically as

$$S = \Psi^{-1} R(z,u) e^{\mathbf{u}/z}, \tag{84}$$

where $R(z,u)$ satisfies properties (i-ii) of Theorem 1.

Since the canonical coordinates $u^i(Q,t)$ defined by materialization vanish at the origin, the product (84) is well-defined. Hence, we may manipulate (84) as a true (rather than formal) object — see the discussion in Section 4.5 of Chapter 1 following the statement of Theorem 1.

Lemma 41 defines a canonical R-calibration for $\mathcal{F}^*_{\mathbf{T}}(X)$. A modified R-calibration will be required for Givental's conjecture in the equivariant case. We will first prove Lemma 41 and study the canonical R-calibration.

Proof. The right side of equation (83) certainly satisfies the quantum differential equation (the non-equivariant case was discussed in Section 5.3 of Chapter 2, see also [**3**], [**6**], [**16**]). Since Ψ^{-1} and $e^{\mathbf{u}/z}$ are well-defined and invertible, R is uniquely determined from the factorization (84). We must prove R has a matrix series expansion in z which satisfies parts (i-ii) of Theorem 1.

First, we will study the following series via fixed point localization:

$$1 + e_j \langle\langle \phi_j, \frac{\phi_j}{z-\psi} \rangle\rangle_0^X. \tag{85}$$

Consider the contribution,

$$C_j = 1 + \sum_{n\geq 0} \sum_{\beta \in E} \sum_{\Gamma \in G^{\overline{u}_j}_{0,2+n}(X,\beta)} \mathrm{Cont}_\Gamma \ e_j \langle\langle \phi_j, \frac{\phi_j}{z-\psi} \rangle\rangle_0^X.$$

Following the notation of Chapter 4, $G^{\overline{u}^j}_{0,2+n}(X,\beta)$ denotes the set of graphs for which markings 1 and 2 lie on a single irreducible component contracted to p_j. The leading constant term is viewed as a degenerate graph contribution of the above type.

By the localization formula and the definitions of $v_1(T,z)$ and A^j, we obtain,

$$C_j = v_1(A^j, z).$$

By Proposition 5 and Lemma 40,

$$v_1(A^j, z) = e^{u(A^j)/z} = e^{u^j/z}.$$

Next, we compute the following series via fixed point localization:

$$\frac{1}{\chi + z} + e_j \langle\langle \frac{\phi_j}{\chi - \psi}, \frac{\phi_j}{z-\psi} \rangle\rangle_0^X. \tag{86}$$

Consider the contribution,

$$C_{j,\chi} = \frac{1}{\chi + z} + \sum_{n\geq 0} \sum_{\beta \in E} \sum_{\Gamma \in G^{\overline{u}^j}_{0,2+n}(X,\beta)} \mathrm{Cont}_\Gamma \ e_j \langle\langle \frac{\phi_j}{\chi - \psi}, \frac{\phi_j}{z-\psi} \rangle\rangle_0^X.$$

The leading polar term is viewed as a degenerate graph contribution.

By the localization formula and the definitions of $v_2(T,x,y)$ and A^j, we obtain,

$$C_{j,\chi} = v_2(A^j, \chi, z).$$

By Proposition 5 and Lemma 40,

$$v_2(A^j,\chi,z) = \frac{e^{u(A^j)/\chi + u(A^j)/z}}{\chi+z} = \frac{e^{u^j/\chi+u^j/z}}{\chi+z}.$$

We now study the matrix elements S_j^{μ}. First assume $\mu = j$. We will prove S_j^{μ} can be expressed in the following form:

$$S_j^j = C_j\sqrt{e_j} + \sum_{\chi\in\mathbf{T}_j^*} D_\chi(Q,t)\ C_{j,\chi}(Q,t,z)\sqrt{e_j}, \tag{87}$$

where the sum in the second term is over the set $\mathbf{T}_j^*$ of torus characters which occur as tangent characters for edges incident to p_j.

Let G be a graph contributing to the integral

$$S_j^j = \sqrt{e_j} + \langle\langle \phi_\mu, \frac{\phi_j}{z-\psi}\rangle\rangle_0^X e_j\sqrt{e_j}.$$

There are two cases:

(i) markings 1 and 2 lie on the same vertex,
(ii) markings 1 and 2 lie on different vertices.

For graphs of type (ii), there is a unique minimal path from the vertex carrying the first marking to the vertex carrying the second marking. An element of $\mathbf{T}_j^*$ is associated to G by the last edge in the minimal path.

We now prove equation (87). The sum of all type (i) contributions to S_j^j is exactly $C_j\sqrt{e_j}$ (accounting also for the constant term). The type (ii) contributions to S_j^j are summed as follows. Let $\chi\in\mathbf{T}_j^*$ be a fixed tangent character. Consider the subsummation of type (ii) contributions for which the associated character is χ. The sum is easily seen to factor as

$$D_\chi(Q,t)\ C_{j,\chi}(Q,t,z)\sqrt{e_j},$$

where the first factor does *not* depend upon z. A summation over χ then yields (87).

The z dependence of S_j^j is now easily determined. Equation (87) and the expansions,

$$C_j = e^{u^j/z},$$

$$C_{j,\chi} = \Big(\sum_k e^{u^j/\chi}\frac{(-z)^k}{\chi^{k+1}}\Big)\ e^{u^j/z},$$

prove S_j^j has a series expansion in positive powers of z (up to the factor $e^{u^j/z}$).

The analysis of S^μ_j for $\mu \neq j$ is identical. If $\mu \neq j$, only type (ii) graphs contribute, and the expression,

$$S^\mu_j = \sum_{\chi \in \mathbf{T}^*_j} D_\chi(Q,t)\ C_{j,\chi}(Q,t,z)\sqrt{e_j},$$

is obtained.

The matrix S is canonically expressed as a product of a matrix series in non-negative powers of z with $e^{\mathbf{u}/z}$. Since Ψ^{-1} has no z dependence, R is a matrix series in z. Part (i) of Theorem 1 is established.

Part (ii) of Theorem 1 is deduced from equation (4) of Chapter 1. The factors of $\sqrt{e_j}$ in the definition of S are inserted to satisfy the unitary condition. $\square$

Double parentheses will be defined to include unstable degree 0 contributions:

$$\begin{aligned} ((\phi_\mu, \frac{\phi_j}{z-\phi_j}))^X_0 &= z\ \langle\langle \phi_\mu, \frac{\phi_j}{z-\phi_j}, \phi_{\mathbf{1}}\rangle\rangle^X_0 \\ &= \frac{\delta^\mu_j}{e_\mu} + \langle\langle \phi_\mu, \frac{\phi_j}{z-\phi_j}\rangle\rangle^X_0 \end{aligned}$$

The formula for S is then simply:

$$S^\mu_j = ((\phi_\mu, \frac{\phi_j}{z-\psi}))^X_0\, e_\mu \sqrt{e_j}. \tag{88}$$

4. The functions E^{ij}_{kl} for the canonical calibration

The functions E^{ij}_{kl} are obtained from the canonical calibration by an expansion of

$$\frac{1}{w+z}\sum_\mu (S^t(w))^i_\mu\ \frac{1}{e_\mu}\ (S(z))^\mu_j.$$

We will study the above expression via localization graph sums.

Define the double parenthesized series below to contain unstable degree 0 contributions:

$$((\frac{\phi_i}{w-\psi}, \frac{\phi_j}{z-\psi}))^X_0 = (\frac{1}{w}+\frac{1}{z})^{-1}\ \langle\langle \frac{\phi_i}{w-\psi}, \frac{\phi_j}{z-\psi}, 1\rangle\rangle^X_0. \tag{89}$$

Lemma 42. *Let S^μ_j be the canonical calibration. Then,*

$$\frac{1}{w+z}\sum_\mu (S^t(w))^i_\mu\ \frac{1}{e_\mu}\ (S(z))^\mu_j = ((\frac{\phi_i}{w-\psi}, \frac{\phi_j}{z-\psi}))^X_0 \sqrt{e_i e_j}.$$

Proof. The Lemma is a straightforward consequence of the WDVV equation:

$$\sum_\mu \langle\langle \phi_\mu, \frac{\phi_i}{w-\psi}, \phi_{\mathbf{1}}\rangle\rangle_0^X \ e_\mu \ \langle\langle \phi_\mu, \frac{\phi_j}{z-\psi}, \phi_{\mathbf{1}}\rangle\rangle_0^X =$$

$$\sum_\mu \langle\langle \frac{\phi_i}{w-\psi}, \frac{\phi_j}{z-\psi}, \phi_\mu\rangle\rangle_0^X \ e_\mu \ \langle\langle \phi_\mu, \phi_{\mathbf{1}}, \phi_{\mathbf{1}}\rangle\rangle_0^X.$$

By (88), the left side is:

$$\frac{1}{\sqrt{e_i e_j}\ wz} \ \sum_\mu (S^t(w))^i_\mu \ \frac{1}{e_\mu} \ (S(z))^\mu_j,$$

By the axiom of the fundamental class and (89), the right side is

$$(\frac{1}{w}+\frac{1}{z}) \ ((\frac{\phi_i}{w-\psi}, \frac{\phi_j}{z-\psi}))_0^X,$$

completing the proof. $\square$

Define the functions E^{ij}_{kl} as coefficients of a sum over localization graphs for which the first marking lies at an end over p_i and the second marking lies at an end over p_j:

- Let $G^{E^{ij}}_{0,2+n}(X,\beta) \subset G_{0,1+n}(X,\beta)$ denote the set of graphs for which the marking 1 lies on a valence two vertex mapped to p_i and the marking 2 lies on a valence two vertex mapped to p_j. Then, $E^{ij}_{kl}(Q,t)$ is defined as the coefficient of $\psi_1^k\psi_2^l$ in

$$\sum_{n\geq 0}\sum_{\beta\in E} \ \sum_{\Gamma\in G^{E^{ij}}_{0,2+n}(X,\beta)} \frac{e^{u^i/\chi_{1,\Gamma}}}{(\chi_{1,\Gamma}-\psi_1)} \frac{e^{u^j/\chi_{2,\Gamma}}}{(\chi_{2,\Gamma}-\psi_2)} \frac{\mathrm{Cont}_\Gamma \ \sqrt{e_i e_j}\langle\phi_i,\phi_j,\gamma^n\rangle}{n!},$$

  where $\chi_{1,\Gamma}, \chi_{2,\Gamma}$ are the torus characters corresponding to the unique edges of Γ incident to the markings 1,2 respectively.

LEMMA 43.

$$\frac{1}{w+z}\sum_\mu (S^t(w))^i_\mu \ \frac{1}{e_\mu} \ (S(z))^\mu_j =$$

$$e^{u^i/w+u^j/z}\big(\frac{\delta_{ij}}{z+w} + \sum_{k,l=0}^{\infty} E^{ij}_{kl}(-w)^k(-z)^l\big).$$

Proof. Let G be a graph contributing to the integral,

$$((\frac{\phi_i}{w-\psi}, \frac{\phi_j}{z-\psi}))_0^X \sqrt{e_i e_j},$$

via fixed point localization. There are two cases:

(i) markings 1 and 2 lie on the same vertex,
(ii) markings 1 and 2 lie on different vertices.

The sum of all type (i) contributions is

$$e^{u^i/w+u^j/z}\frac{\delta_{ij}}{w+z},$$

as obtained in the proof of Lemma 41.

We will prove the sum of all type (ii) contributions is

$$e^{u^i/w+u^j/z}\sum_{k,l=0}^{\infty} E^{ij}_{kl}(-w)^k(-z)^l. \tag{90}$$

Each graph of G of type (ii) can be partitioned uniquely as left, central, and right subgraphs:

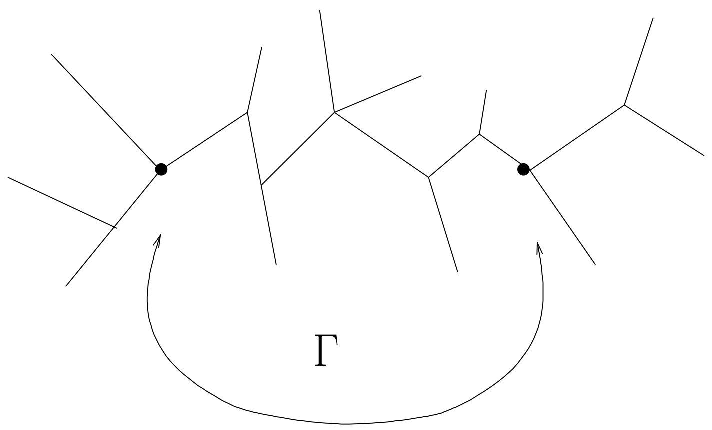

Figure 7

The central graph Γ is canonically an element of the set $G^{E^{ij}}_{0,2+n}(X,\beta)$ for appropriate n and β. The contributions of all graphs G of type (ii) with a fixed central graph Γ may be summed by applying Proposition 5 to the left and right subgraphs:

$$\frac{e^{u^i/w+u^i/\chi_{1,\Gamma}}}{(w+\chi_{1,\Gamma})}\frac{e^{u^j/z+u^j/\chi_{2,\Gamma}}}{(z+\chi_{2,\Gamma})}\frac{\mathrm{Cont}_\Gamma\ \sqrt{e_ie_j}\langle\phi_i,\phi_j,\gamma^n\rangle}{n!}, \tag{91}$$

using the notation for $\Gamma \in G^{E^{ij}}_{0,2+n}(X,\beta)$ introduced above. Then, after summing (91) over the set $G^{E^{ij}}_{0,2+n}(X,\beta)$ of all central graphs, we obtain (90). $\square$

By Lemma 43, the functions $E^{kl}_{ij}(q,t)$ defined by localization graph sums equal the eponymous functions canonically associated to the canonical calibration S by the definitions of Chapter 6.

5. The functions T^i_k for the canonical R-calibration

The functions T^i_k are obtained from the canonical R-calibration by an expansion of $S^{\mathbf{1}}_i$. By (88), we find

$$S^{\mathbf{1}}_i = ((\phi_{\mathbf{1}}, \frac{\phi_i}{z-\psi}))\sqrt{e_i}.$$

We will express $S^{\mathbf{1}}_i$ via localization graph sums.

Let $\mathbf{Q}$ denote the variable set $\{Q_k\}_{k\geq 0}$, and as before, let

$$Q(\psi) = \sum_{k=0}^{\infty} Q_k \psi^k$$

Define the function $f_i(\mathbf{Q})$ by:

$$f_i(\mathbf{Q}) = \frac{1}{\sqrt{e_i}} + \frac{Q(-z)}{\sqrt{e_i}z} + \frac{1}{\sqrt{e_i}z}\sum_{n\geq 2}\frac{1}{n!}\int_{\overline{M}_{0,n+1}} Q(\psi_1)\ldots Q(\psi_n) \cdot \frac{1}{z-\psi_{n+1}}. \tag{92}$$

Lemma 44. *$S^{\mathbf{1}}_i$ is obtained by evaluation of f_i,*

$$S^{\mathbf{1}}_i = \left[f_i(\mathbf{Q})\right]_{Q_k = A^i_k},$$

where $A^i_k(Q,t)$ are the graph sums defined in Section 2.

Proof. We may express $S^{\mathbf{1}}_i$ as:

$$\begin{aligned} S^{\mathbf{1}}_i &= \frac{1}{\sqrt{e_j}} + \langle\langle \phi_{\mathbf{1}}, \frac{\phi_i}{z-\psi}\rangle\rangle^X_0 \sqrt{e_i} \\ &= \frac{1}{\sqrt{e_j}} + \frac{t^i}{\sqrt{e_j}z} + \langle\langle \frac{\phi_i}{z-\psi}\rangle\rangle^X_0 \sqrt{e_j}. \end{aligned}$$

We will analyze the series $\langle\langle \frac{\phi_i}{z-\psi}\rangle\rangle^X_0 \sqrt{e_j}$ by localization.

Let G be localization graphs contributing to $\langle\langle \frac{\phi_i}{z-\psi}\rangle\rangle^X_0 \sqrt{e_j}$. There are two cases:

(i) marking 1 lies on a vertex of valence 2,

(ii) marking 1 lies on a vertex of valence at least 3.

A direct check via the localization formula shows that the term $\frac{t^i}{\sqrt{e_j}z}$ and the type (i) contributions together sum exactly to $\frac{A^i(-z)}{\sqrt{e_j}z}$. The type (ii) contributions exactly yield the third summand of $f_i(\mathbf{Q})$ evaluated at $Q_k = A^i_k$. $\square$

The functions A^i define coordinates of a point of the manifold $\mathcal{T}$:

$$p = (A^i_0, A^i_1, A^i_2, \ldots) \in \mathcal{T}.$$

By Section 6 of Chapter 5, there is a canonical point

$$r = (T^i_0, T^i_1, T^i_2, \ldots) \in \mathcal{T}$$

obtained by the string and dilaton flows satisfying $T^i_0 = T^i_1 = 0$. The coordinates T^i_k are series in Q, t.

Lemma 45. *We have*

$$S^{\mathbf{1}}_i(z) = \frac{e^{u^i/z}}{\sqrt{\Delta^i}} \left(1 - \sum_{k=0}^{\infty} T^i_k (-z)^{k-1} \right).$$

Proof. We consider $f_i(\mathbf{Q})$ as a function on $\mathcal{T}$ with coordinates given by the variables $\mathbf{Q}$. It is important to distinguish the functions $u, \sqrt{\Delta}$ on $\mathcal{T}$ from the functions $u^i, \sqrt{\Delta^i}$ of the Frobenius manifold.

The string operator on $\mathcal{T}$ is:

$$\mathcal{L} = \partial_0 - \sum_{k=0}^{\infty} Q^i_{k+1} \partial_k.$$

A direct check yields:

$$\mathcal{L} f_i = z f_i. \tag{93}$$

Let q be obtained from p by following the string flow for time $-u(p)$. Then, by (93),

$$f_i(q) = e^{-u(p)/z} f_i(p).$$

The dilaton operator on $\mathcal{T}$ is:

$$\mathcal{D} = \partial_0 - \sum_{k=0}^{\infty} Q^i_k \partial_k.$$

A direct check yields:

$$\mathcal{D} f_i = -f_i. \tag{94}$$

Let r be obtained from q by following the dilaton flow for time $-\log \sqrt{\Delta}(p)$. Then, by (94),

$$f_i(r) = e^{\log \sqrt{\Delta}(p)} f_i(q).$$

We conclude

$$S_i^{\mathbf{1}} = \frac{e^{u(p)/z}}{\sqrt{\Delta}(p)} f_i(r). \tag{95}$$

We now analyze the right side of (95). Since $r_0 = r_1 = 0$, the third summand of $f_i(r)$ vanishes for dimension reasons. Hence,

$$f_i(r) = \frac{1}{\sqrt{e_j}} \left(1 - \sum_{k=0}^{\infty} T_k^i (-z)^{k-1} \right).$$

Then, by Lemma 40, we obtain:

$$S_i^{\mathbf{1}} = \frac{e^{u^i/z}}{\sqrt{\Delta^i}} \left(1 - \sum_{k=0}^{\infty} T_k^i (-z)^{k-1} \right),$$

completing the derivation. $\square$

By Lemma 45, the functions $T_k^i(Q, t)$ defined by flows from A_k^i equal the eponymous functions canonically associated to the canonical calibration S by the definitions of Chapter 6.

6. Modified R-calibrations

Let S be the canonical fundamental solution for $\mathcal{F}_{\mathbf{T}}^*(X)$ obtained from equivariant Gromov-Witten theory in Lemma 41. All fundamental solutions satisfying part (i) and (ii) of Theorem 1 are of the form

$$S \ \exp(\sum_{k\geq 1} \mathbf{a}_{2k-1} z^{2k-1}),$$

where

$$\{ \ \mathbf{a}_{2k-1} = \mathrm{Diag}(a^1_{1,2k-1}, a^2_{2,2k-1}, \cdots, a^m_{m,2k-1}) \ \}$$

are constant diagonal matrices by part (iii) of Theorem 1.

The Bernoulli numbers B_m are defined by the following series:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

We will require a canonical modification of S,

$$\bar{S} = S \ \exp(\sum_{k\geq 1} \mathbf{a}_{2k-1} z^{2k-1}),$$

where the constant diagonal matrices are defined by

$$a^i_{i,2k-1} = -N_{2k-1}(\frac{1}{\chi^i}) \frac{B_{2k}}{(2k-1)(2k)}.$$

Here, N_r denotes the r^{th} Newton sum.

The canonical square roots $\sqrt{\Delta^i}$ and $\bar{S}$ determine a modified calibration. The functions $\bar{R}$, $\bar{E}^{ij}_{kl}$, and $\bar{T}^i_k$ are obtained from the modified calibration by the definitions of Chapter 6.

THEOREM 2. *Givental's conjecture for equivariant Gromov-Witten theory holds for the modified fundamental solution $\bar{S}$:*

(i) *genus 1,*

$$dF^X_{1,\mathbf{T}} = \sum_i \frac{1}{48} d(\log \Delta^i) + \frac{1}{2}(\bar{R}_1)^i_i du^i,$$

(ii) *higher genus,*

$$\exp\left(\sum_{g\geq 2} \lambda^{g-1} F^X_{g,\mathbf{T}}\right) =$$

$$\left[\exp\left(\frac{\lambda}{2}\sum_{k,l=0}^{\infty}\sum_{i,j}\bar{E}^{ij}_{kl}\sqrt{\Delta^i}\sqrt{\Delta^j}\frac{\partial}{\partial Q^i_k}\frac{\partial}{\partial Q^j_l}\right)\prod_m \tau(\lambda\Delta^m, Q^m)\right]_{Q^i_k=\bar{T}^i_k}.$$

Givental's proof of Theorem 2 via localization graphs and Hodge integral techniques will be given in Parts 3.

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